BOLIB 2019: BILEVEL OPTIMIZATION LIBRARY OF TEST PROBLEMS VERSION 2

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ABSTRACT. To help accelerate the development of numerical solvers for bilevel optimization, BOLIB aims at presenting a collection of academic and real-world applications or case studies on the problem. The original version, with corresponding MATLAB source files are available at https://arxiv.org/abs/1812.00230, is made of 124 academic examples of nonlinear bilevel optimization problems. In this version, we collect 138 nonlinear (including 124 nonlinear ones from the first version), 24 linear and 11 simple bilevel optimization problems. This again is the first time that such a scale of examples are provided to render a uniform basis on which algorithms proposed to deal with bilevel optimization can be tested and compared. All the collected examples are programmed via Matlab and the library will be made freely available online.

CONTENTS

1. INTRODUCTION

The general bilevel optimization problem can take the form

$$\begin{aligned} \min_{x,y} \quad & F(x,y) \\ \text{s.t.} \quad & G(x,y) \le 0,\ H(x,y) = 0, \\ & y \in S(x) := \arg\min_y \{f(x,y) :\ g(x,y) \le 0,\ h(x,y) = 0\}, \end{aligned} \tag{1.1}$$

where the functions $G : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}^{n_G}$ and $H : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}^{n_H}$ define the upper-level constraints, while $g : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}^{n_g}$ and $h : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}^{n_h}$ describe the lower-level

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constraints. On the other hand, $F : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}$ and $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}$ denote the upper-and lower-level objective/cost functions, respectively. The set-valued map $S : \mathbb{R}^{n_x} \rightrightarrows \mathbb{R}^{n_y}$ represents the optimal solution/argminimum mapping of the lower-level problem. Further recall that problem (1.1) as a whole is often called upper-level problem.

As the medium and long term goal of this library is to include various classes of bilevel optimization problems, in particular, simple, linear, nonlinear, and real-world applications or case studies, we intentionally consider our model (1.1) to be broad, as it is likely that the overwhelming majority of these problems will be of this form. Recall that problem (1.1) is linear if all the functions involved are linear; otherwise, it is nonlinear.

Another important restriction made in the codes of this version of the library is that equality constraint are not considered. This is because in the test set presented here, only four examples have equality constraints; see Table 4 or the detailed formulas of the problems in Appendix 3. Hence, we use the fact that $H(x, y) = 0$ (similarly to $h(x, y) = 0$) can be expressed as $H(x, y) \leq 0$ and $-H(x, y) \leq 0$. Changes will be introduced in future versions of the library, provided that a significant number of new examples include equality constraints in the upper or lower-level of the problem. Hence, our focus here will be on nonlinear bilevel optimization problems of the form

$$\begin{aligned} \min_{x,y} \quad & F(x, y) \\ \text{s.t.} \quad & G(x, y) \leq 0, \\ & y \in S(x) := \arg\min_{y} \{f(x, y) : \; g(x, y) \leq 0\}. \end{aligned} \tag{1.2}$$

For the sake of clearness, we classify bilevel optimization problems into 3 categorises: Nonlinear, linear, simple ones.

- *Nonlinear bilevel programs* at least have one of the functions in (1.2) being nonlinear.
- *Linear bilevel programs* have all functions being linear.
- *Simple bilevel programs* are optimization problems where the optimal point for a given function is being sought from a set partly defined by the optimal solution set of a second optimization problem. But unlike in (1.2), the lower-level problem is not a parametric optimization problem. More precisely, a simple bilevel optimization has the form:

$$\begin{aligned} \min_{y} \quad & F(y) \\ \text{s.t.} \quad & G(y) \leq 0, \\ & y \in S := \arg\min_{y} \{f(y) : \; g(y) \leq 0\}. \end{aligned} \tag{1.3}$$

  where the function $G : \mathbb{R}^{n_y} \to \mathbb{R}^{n_G}$ defines the upper-level constraints, while $g : \mathbb{R}^{n_y} \to \mathbb{R}^{n_g}$ describes the lower-level constraints. Similarly to (1.2), $F : \mathbb{R}^{n_y} \to \mathbb{R}$ and $f : \mathbb{R}^{n_y} \to \mathbb{R}$ denote the upper-and lower-level objective/cost functions, respectively. Examples of papers where simple bilevel programs are investigated include [13]. Jane Ye and her co-autors, see, e.g., [32], use the expression "simple bilevel programs" for bilevel optimization problems of the form (1.2), where y is not involved in the upper-level constraints and x is not part of the lower-level constraints.

This paper provides a unique platform for the development of numerical methods, as well as theoretical results for bilevel optimization problems. The main contributions of the paper can be summarized as follows:

(1) This first version of BOLIB provides codes for 173 examples including 138 nonlinear, 24 linear and 11 simple bilevel optimization problems, ready to be used to test numerical algorithms.

(2) BOLIB provides the true or best known solutions and the corresponding references for all the examples included in the library. Hence, can therefore serve as a benchmark for numerical accuracy for methods designed to solve (1.2).

(3) All the mathematical formulas of the examples present in the library are also given, in order to allow researchers to test theoretical properties or build codes for the examples in software different from MATLAB, if necessary. For each example, the formulas of the functions F, G, f, and g, involved in (1.2), are put in Section 3, as well as some useful background details.

To the best of our knowledge, this is the largest library of test examples for bilevel optimization, especially for the nonlinear class of the problem. It includes problems from Benoit Colson's BIPA [11], Sven Leyffer's MacMPEC [35], as well as from Mitsos and Barton's technical report [39]. Special classes of examples from the latter test sets not in this version of BOLIB will be included in future versions, where corresponding classes of problems are expanded to a reasonable size.

The main goal that we hope to achieve with BOLIB is the acceleration of numerical software development for bilevel optimization, as it is our opinion that the level of expansion of applications of the problem has outpaced the development rate for numerical solvers, especially for the nonlinear class of the problem.

2. Descriptions of Library

This section mainly describes inputs and outputs of each example, and also lists all examples, together with their true or best known solutions and the corresponding references. Before we proceed, note that each `m-file` contains information about the corresponding example, which include the first and second order derivatives of the input functions. For the upper-level objective function $F : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \longrightarrow \mathbb{R}$, these derivatives are defined as follows

$$\begin{aligned}
\nabla_x F(x,y) &= \begin{bmatrix} \nabla_{x_1} F \\ \vdots \\ \nabla_{x_{n_x}} F \end{bmatrix} \in \mathbb{R}^{n_x}, \\
\nabla^2_{xx} F(x,y) &= \begin{bmatrix} \nabla^2_{x_1 x_1} F & \cdots & \nabla^2_{x_{n_x} x_1} F \\ \vdots & \ddots & \vdots \\ \nabla^2_{x_1 x_{n_x}} F & \cdots & \nabla^2_{x_{n_x} x_{n_x}} F \end{bmatrix} \in \mathbb{R}^{n_x \times n_x}, \\
\nabla^2_{xy} F(x,y) &= \begin{bmatrix} \nabla^2_{x_1 y_1} F & \cdots & \nabla^2_{x_{n_x} y_1} F \\ \vdots & \ddots & \vdots \\ \nabla^2_{x_1 y_{n_x}} F & \cdots & \nabla^2_{x_{n_x} y_{n_x}} F \end{bmatrix} \in \mathbb{R}^{n_y \times n_x}.
\end{aligned} \tag{2.1}$$

Similar expressions are valid for $\nabla_y F(x,y) \in \mathbb{R}^{n_y}$, $\nabla^2_{yy} F(x,y) \in \mathbb{R}^{n_y \times n_y}$ and the lower-level objective function f. As the constraint functions are vector-valued, we have for instance, in the context of the upper-level constraint function $G : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \longrightarrow \mathbb{R}^{n_G}$, that

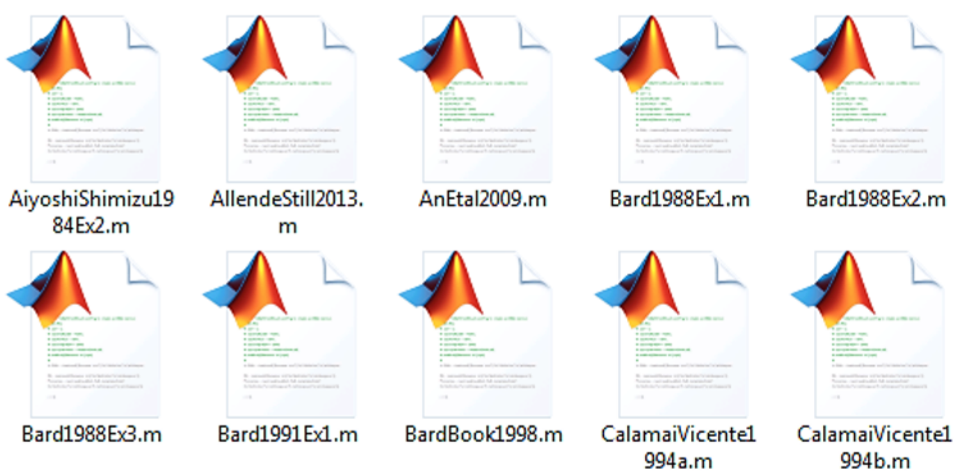
$$\nabla_x G(x,y) = \begin{bmatrix} \nabla_x G_1 \\ \vdots \\ \nabla_x G_{n_G} \end{bmatrix} = \begin{bmatrix} \nabla_{x_1} G_1 & \cdots & \nabla_{x_{n_x}} G_1 \\ \vdots & \ddots & \vdots \\ \nabla_{x_1} G_{n_G} & \cdots & \nabla_{x_{n_x}} G_{n_G} \end{bmatrix} \in \mathbb{R}^{n_G \times n_x}, \tag{2.2}$$

$$\nabla^2_{xx}G(x,y) = \begin{bmatrix} \nabla^2_{xx}G_1 \\ \vdots \\ \nabla^2_{xx}G_{n_G} \end{bmatrix} = \begin{bmatrix} \nabla^2_{x_1x_1}G_1 & \cdots & \nabla^2_{x_{n_x}x_1}G_1 \\ \vdots & \ddots & \vdots \\ \nabla^2_{x_1x_{n_x}}G_1 & \cdots & \nabla^2_{x_{n_x}x_{n_x}}G_1 \\ \vdots & \ddots & \vdots \\ \nabla^2_{x_1x_1}G_{n_G} & \cdots & \nabla^2_{x_{n_x}x_1}G_{n_G} \\ \vdots & \ddots & \vdots \\ \nabla^2_{x_1x_{n_x}}G_{n_G} & \cdots & \nabla^2_{x_{n_x}x_{n_x}}G_{n_G} \end{bmatrix} \in \mathbb{R}^{(n_Gn_x)\times n_x}, \quad (2.3)$$

$$\nabla^2_{xy}G(x,y) = \begin{bmatrix} \nabla^2_{xy}G_1 \\ \vdots \\ \nabla^2_{xy}G_{n_G} \end{bmatrix} = \begin{bmatrix} \nabla^2_{x_1y_1}G_1 & \cdots & \nabla^2_{x_{n_x}y_1}G_1 \\ \vdots & \ddots & \vdots \\ \nabla^2_{x_1y_{n_y}}G_1 & \cdots & \nabla^2_{x_{n_x}y_{n_y}}G_1 \\ \vdots & \ddots & \vdots \\ \nabla^2_{x_1y_1}G_{n_G} & \cdots & \nabla^2_{x_{n_x}y_1}G_{n_G} \\ \vdots & \ddots & \vdots \\ \nabla^2_{x_1y_{n_y}}G_{n_G} & \cdots & \nabla^2_{x_{n_x}y_{n_y}}G_{n_G} \end{bmatrix} \in \mathbb{R}^{(n_Gn_y)\times n_x}. \quad (2.4)$$

Similar formulas will be valid for $\nabla_y G(x,y) \in \mathbb{R}^{n_G\times n_y}$, $\nabla^2_{yy}G(x,y) \in \mathbb{R}^{n_Gn_y\times n_y}$ and the lower-level constraint g. Here we need to emphasize that when $n_G = 1$, $\nabla_x G(x,y) \in \mathbb{R}^{1\times n_x}$ which is a row vector whilst $\nabla_x F(x,y) \in \mathbb{R}^{n_x}$ or $\nabla_x f(x,y) \in \mathbb{R}^{n_x}$ are column vectors.

2.1. **Inputs and outputs.** Open folder `BOLIBExample` which at least contains 3 sub-folders. They are `Nonlinear`, `Linear` and `Simple`. In folder `Nonlinear`, there are 138 MATLAB m-files. Each one specifies a nonliner bilevel test example, named by a combination of authors' surnames, year of publication, and when necessary, the order of the example in the corresponding reference. For example, as in following figure (showing a partial list of the examples), `AiyoshiShimizu1984Ex2.m` stands for the Example 2 considered by Aiyoshi and Shimizu in 1984, see [1] for more details.



In folder `Linear`, there are 24 MATLAB m-files defining 24 liner bilevel test examples. The rule of naming each example is same as above. Similarly, folder `Simple` contains 11 simple bilevel test examples.

Now we describe the inputs and outputs for a given m-file example. All files have the uniform function handle as

```
w=example_name(x,y,keyf,keyxy).
```
(2.5)

For the inputs, we have

$$\begin{aligned} x &\in \mathbb{R}^{n_x}, \;\; y \in \mathbb{R}^{n_y}, \\ \texttt{keyf} &\in \{\texttt{'F'}, \texttt{'G'}, \texttt{'f'}, \texttt{'g'}\}, \\ \texttt{keyxy} &\in \{[\,], \texttt{'x'}, \texttt{'y'}, \texttt{'xx'}, \texttt{'xy'}, \texttt{'yy'}\}, \end{aligned}$$

where `'F'`, `'G'`, `'f'`, and `'g'` respectively stand for the four functions involved in (1.2). `'x'` and `'y'` represent the first order derivative with respect to x and y, respectively. Finally, `'xx'`, `'xy'`, and `'yy'` correspond to the second order derivative of the function F, G, f, and g, with respect to xx, xy, and yy, respectively.

For the outputs, $w = \texttt{example\_name}(x, y, \texttt{keyf})$ or $w = \texttt{example\_name}(x, y, \texttt{keyf}, [\,])$ returns the function value of `keyf`, and $w = \texttt{example\_name}(x, y, \texttt{keyf}, \texttt{keyxy})$ returns the first or second order derivative of `keyf` with respect to choice of `keyxy` as describe above. We can summarize the input-inputs scenarios in the following table:

keyfkeyxy	[ ]	'x'	'y'	'xx'	'xy'	'yy'
'F'	$F(x,y)$	$\nabla_x F(x,y)$	$\nabla_y F(x,y)$	$\nabla^2_{xx} F(x,y)$	$\nabla^2_{xy} F(x,y)$	$\nabla^2_{yy} F(x,y)$
'G'	$G(x,y)$	$\nabla_x G(x,y)$	$\nabla_y G(x,y)$	$\nabla^2_{xx} G(x,y)$	$\nabla^2_{xy} G(x,y)$	$\nabla^2_{yy} G(x,y)$
'f'	$f(x,y)$	$\nabla_x f(x,y)$	$\nabla_y f(x,y)$	$\nabla^2_{xx} f(x,y)$	$\nabla^2_{xy} f(x,y)$	$\nabla^2_{yy} f(x,y)$
'g'	$g(x,y)$	$\nabla_x g(x,y)$	$\nabla_y g(x,y)$	$\nabla^2_{xx} g(x,y)$	$\nabla^2_{xy} g(x,y)$	$\nabla^2_{yy} g(x,y)$

For the dimension of w in each scenario, see (2.1)–(2.4). If $n_G = 0$ (or $n_g = 0$), all outputs related to G (or g) should be empty, namely, $w = [\,]$. Let us look at some specific usage:

- $w = \texttt{example\_name}(x, y, \texttt{'F'})$ or $w = \texttt{example\_name}(x, y, \texttt{'F'}, [\,])$ returns the function value of F, i.e., $w = F(x,y)$; this is similar for G, f, and g;
- $w = \texttt{example\_name}(x, y, \texttt{'F'}, \texttt{'x'})$ returns the partial derivative of F with respect to x, i.e., $w = \nabla_x F(x,y)$;
- $w = \texttt{example\_name}(x, y, \texttt{'G'}, \texttt{'y'})$ returns the Jacobian matrix of G with respect to y, i.e., $w = \nabla_y G(x,y)$;
- $w = \texttt{example\_name}(x, y, \texttt{'f'}, \texttt{'xy'})$ returns the Hessian matrix of f with respect to xy, i.e., $w = \nabla^2_{xy} f(x,y)$;
- $w = \texttt{example\_name}(x, y, \texttt{'g'}, \texttt{'yy'})$ returns the second order derivative of g with respect to yy, i.e., $w = \nabla^2_{yy} g(x,y)$.

We now use two examples to illustrate the definitions above. The first one is a nonlinear test example and the second one is a simple bilevel test example.

Example 2.1. *Shimizu et al. (1997), see* [50]*, considered the bilevel program* (1.2) *with*

$$\begin{aligned} F(x,y) &:= (x-5)^2 + (2y+1)^2, \\ f(x,y) &:= (y-1)^2 - 1.5xy, \\ g(x,y) &:= \begin{bmatrix} -3x + y + 3 \\ x - 0.5y - 4 \\ x + y - 7 \end{bmatrix}. \end{aligned}$$

Clearly, $n_x = 1, n_y = 1, n_G = 0, n_g = 3$. *The* `m-file` *is named by* `ShimizuEtal1997a` *(i.e.,* `exmaple_name = ShimizuEtal1997a` *), which was coded through MATLAB as follows.*

```
function w=ShimizuEtal1997a(x,y,keyf,keyxy)
if nargin<4 || isempty(keyxy)
    switch keyf
    case 'F'; w = (x-5)^2+(2*y+1)^2;
    case 'G'; w = [];
    case 'f'; w = (y-1)^2-1.5*x*y;
    case 'g'; w = [-3*x+y+3; x-0.5*y-4; x+y-7];
    end
else
    switch keyf
    case 'F'
        switch keyxy
        case 'x' ; w = 2*(x-5);
        case 'y' ; w = 4*(2*y+1);
        case 'xx'; w = 2;
        case 'xy'; w = 0;
        case 'yy'; w = 8;
        end
    case 'G'
       switch keyxy
        case 'x' ; w = [];
        case 'y' ; w = [];
        case 'xx'; w = [];
        case 'xy'; w = [];
        case 'yy'; w = [];
        end
    case 'f'
        switch keyxy
        case 'x' ; w = -1.5*y;
        case 'y' ; w = 2*(y-1)-1.5*x;
        case 'xx'; w = 0;
        case 'xy'; w = -1.5;
        case 'yy'; w = 2;
        end
    case 'g'
        switch keyxy
        case 'x' ; w = [-3;  1;  1];
        case 'y' ; w = [ 1;-0.5; 1];
        case 'xx'; w = [ 0;  0;  0];
        case 'xy'; w = [ 0;  0;  0];
        case 'yy'; w = [ 0;  0;  0];
        end
     end
end
end
```

If we are given some inputs (as in left column of the table below), then `ShimizuEtal1997a` will return us corresponding results as in the right column of the table:

Inputs:		Outputs:	
`x`	`= 4`	`x`	`= 4`
`y`	`= 0`	`y`	`= 0`
`F`	`= ShimizuEtal1997a(x,y,'F')`	`F`	`= 2`
`Fx`	`= ShimizuEtal1997a(x,y,'F','x')`	`Fx`	`= -2`
`Gy`	`= ShimizuEtal1997a(x,y,'G','y')`	`Gy`	`= []`
`fxy`	`= ShimizuEtal1997a(x,y,'f','xy')`	`fxy`	`= -1.5`
`gyy`	`= ShimizuEtal1997a(x,y,'g','yy')`	`gyy`	`= [0;0;0]`

Example 2.2. *Franke et al. (2018), see* [73]*, considered the bilevel program* (1.2) *with*

$$\begin{aligned} F(y) &:= -y_2 \\ f(y) &:= y_3 \\ g(y) &:= \begin{bmatrix} y_1^2 - y_3 \\ y_1^2 + y_2^2 - 1 \\ -y_3 \end{bmatrix} \end{aligned}$$

Clearly, $n_x = 0, n_y = 3, n_G = 0, n_g = 3$*. The* `m-file` *is named by* `FrankeEtal2018Ex513` *(i.e.,* `exmaple_name` = `FrankeEtal2018Ex513` *), which was coded through MATLAB as follows.*

```
function w=FrankeEtal2018Ex513(x,y,keyf,keyxy)
if nargin<4 || isempty(keyxy)
    switch keyf
    case 'F'; w = -y(2);
    case 'G'; w = [];
    case 'f'; w = y(3);
    case 'g'; w = [y(1)^2-y(3); y(1)^2+y(2)^2-1; -y(3)];
    end
else
    switch keyf
    case 'F'
        switch keyxy
        case 'x' ; w = 0;
        case 'y' ; w = [0; -1; 0];
        case 'xx'; w = 0;
        case 'xy'; w = zeros(3,1);
        case 'yy'; w = zeros(3,3);
        end
    case 'G'
       switch keyxy
        case 'x' ; w = [];
        case 'y' ; w = [];
        case 'xx'; w = [];
        case 'xy'; w = [];
        case 'yy'; w = [];
        end
    case 'f'
        switch keyxy
        case 'x' ; w = 0;
        case 'y' ; w = [0; 0; 1];
        case 'xx'; w = 0;
        case 'xy'; w = zeros(3,1);
```

```
        case 'yy'; w = zeros(3,3);
        end
    case 'g'
        switch keyxy
        case 'x' ; w = zeros(3,1);
        case 'y' ; w = [2*y(1) 0 -1; 2*y(1) 2*y(2) 0; 0 0 -1];
        case 'xx'; w = zeros(3,1);
        case 'xy'; w = zeros(9,1);
        case 'yy'; w = [2 0 0;0 0 0;0 0 0;2 0 0; 0 2 0;zeros(4,3)];
        end
     end
end
end
```

It is worth mentioning that despite the lack of variable x in Example 2.2, for the sake of unifying the inputs of the function handle as in (2.5), we still treat it as an input. Here, for all simple bilevel examples, we input `x` as a scalar. In this way, `x` has no impact on the example itself.

2.2. **List of test examples.** The details related to each example presented in the BOLIB library are in a column of Table 4 below. As we mentioned before, those examples are classified into 3 categorises: nonliner, linear and simple bilevel optimisation test examples. The first column of the table provides the list of problems, as they appear in the `BOLIBExample` folder. The second column gives the reference in the literature where the example might have first appeared. The third column combines the labels corresponding to the nature of the functions involved in (1.2). Precisely, "N" and "L" will be used to indicate whether the functions F, G, f, and g are nonlinear (N) or linear (L), while "O" is used to symbolize that there is either no function G or g present in problem (1.2). Then follows the column with n_x and n_y for the upper and lower-level variables dimensions, as well as n_G (resp. n_g) to denote the the number of components of the upper (resp. lower)-level constraint function. On the other hand, F^* and f^* denote the best known optimal upper and lower-level objective function values, respectively, according to the literature that is listed in the last column **RefII**.

Note that examples `Zlobec2001b` and `MitsosBartonEx32` have no optimal solutions. There are 4 examples involve parameters. They are `CalamaiVicente1994a` with $\rho \geq 1$ (its F^* and f^* listed in the table are under $\rho = 1$, other cases can be found in Section 3), `HenrionSurowiec2011` with $c \in \mathbb{R}$, `IshizukaAiyoshi1992a` with $M > 1$ and `RobustPortfolioP1` with $\delta \in [1, +\infty]$(its F^* and f^* listed in the table are under $\delta = 2$). Dimensions n_x, n_y, n_G or n_g of examples `OptimalControl`, `RobustPortfolioP1`, `RobustPortfolioP2` and `ShehuEtal2019Ex42` can be altered.

TABLE 4. List of bilevel programs with their labels.

Example name	**RefI**	F-G-f-g	n_x	n_y	n_G	n_g	F^*	f^*	**RefII**
				Nonlinear bilevel examples					
AiyoshiShimizu1984Ex2	[1]	L-L-N-L	2	2	5	6	5	0	[1]
AllendeStill2013	[2]	N-L-N-N	2	2	5	2	1	-0.5	[2]
AnEtal2009	[3]	N-L-N-L	2	2	6	4	2251.6	565.8	[3]
Bard1988Ex1	[4]	N-L-N-L	1	1	1	4	17	1	[4]
Bard1988Ex2	[4]	N-L-N-L	4	4	9	12	-6600	54	[11]
Bard1988Ex3	[4]	N-N-N-N	2	2	3	4	-12.68	-1.02	[8]
Bard1991Ex1	[5]	L-L-N-L	1	2	2	3	2	12	[5]
BardBook1998	[6]	N-L-L-L	2	2	4	7	0	5	
CalamaiVicente1994a	[7]	N-O-N-L	1	1	0	3	0	0	[7]

CalamaiVicente1994b	[7]	N-O-N-L	4	2	0	6	0.3125	-0.4063	[7]
CalamaiVicente1994c	[7]	N-O-N-L	4	2	0	6	0.3125	-0.4063	[7]
CalveteGale1999P1	[9]	L-L-L-N	2	3	2	6	-29.2	0.31	[9, 23]
ClarkWesterberg1990a	[10]	N-L-N-L	1	1	2	3	5	4	[47]
Colson2002BIPA1	[11]	N-L-N-L	1	1	3	3	250	0	
Colson2002BIPA2	[11]	N-L-N-L	1	1	1	4	17	2	[8]
Colson2002BIPA3	[11]	N-L-N-L	1	1	2	2	2	24.02	[8]
Colson2002BIPA4	[11]	N-L-N-L	1	1	2	2	88.79	-0.77	[8]
Colson2002BIPA5	[11]	N-L-N-N	1	2	1	6	2.75	0.57	[8]
Dempe1992a	[12]	L-N-N-N	2	2	1	2	×	×	
Dempe1992b	[12]	N-O-N-N	1	1	0	1	31.25	4	[8]
DempeDutta2012Ex24	[14]	N-O-N-N	1	1	0	1	0	0	[14]
DempeDutta2012Ex31	[14]	L-N-N-N	2	2	4	2	-1	4	[14]
DempeEtal2012	[15]	L-L-N-L	1	1	2	2	-1	-1	[15]
DempeFranke2011Ex41	[16]	N-L-N-L	2	2	4	4	5	-2	[16]
DempeFranke2011Ex42	[16]	N-L-N-L	2	2	4	3	2.13	-3.5	[16]
DempeFranke2014Ex38	[17]	L-L-N-L	2	2	4	4	-1	-4	[17]
DempeLohse2011Ex31a	[18]	N-O-N-L	2	2	0	4	-5.5	0	[18]
DempeLohse2011Ex31b	[18]	N-O-N-L	3	3	0	5	-12	0	
DeSilva1978	[19]	N-O-N-L	2	2	0	4	-1	0	[8]
FalkLiu1995	[20]	N-O-N-L	2	2	0	4	-2.1962	0	[8]
FloudasEtal2013	[21]	L-L-N-L	2	2	4	7	0	200	[52]
FloudasZlobec1998	[22]	N-L-L-N	1	2	2	6	1	-1	[23, 39]
GumusFloudas2001Ex1	[23]	N-L-N-L	1	1	3	3	2250	197.75	[39]
GumusFloudas2001Ex3	[23]	L-L-N-L	2	3	4	9	-29.2	0.31	[39]
GumusFloudas2001Ex4	[23]	N-L-N-L	1	1	5	2	9	0	[39]
GumusFloudas2001Ex5	[23]	L-L-N-N	1	2	2	6	0.19	-7.23	[39]
HatzEtal2013	[24]	L-O-N-L	1	2	0	2	0	0	[24]
HendersonQuandt1958	[25]	N-L-N-L	1	1	2	1	-3266.7	-711.11	[25]
HenrionSurowiec2011	[26]	N-O-N-O	1	1	0	0	$-c^2/4$	$-c^2/8$	[27]
IshizukaAiyoshi1992a	[28]	N-L-L-L	1	2	1	5	0	-M	[28]
KleniatiAdjiman2014Ex3	[29]	L-L-N-L	1	1	2	2	-1	0	[29]
KleniatiAdjiman2014Ex4	[29]	N-N-N-N	5	5	13	11	-10	-3.1	[29]
LamparSagrat2017Ex23	[30]	L-L-N-L	1	2	2	2	-1	1	[30]
LamparSagrat2017Ex31	[31]	N-L-L-L	1	1	1	1	1	0	[31]
LamparSagrat2017Ex32	[31]	N-O-N-O	1	1	0	0	0.5	0	[31]
LamparSagrat2017Ex33	[31]	N-L-L-L	1	2	1	3	0.5	0	[31]
LamparSagrat2017Ex35	[31]	N-L-L-L	1	1	2	3	0.8	-0.4	[31]
LucchettiEtal1987	[33]	N-L-N-L	1	1	2	2	0	0	[33]
LuDebSinha2016a	[34]	N-L-N-O	1	1	4	0	1.14	1.69	[34]
LuDebSinha2016b	[34]	N-L-N-O	1	1	4	0	0	1.66	[34]
LuDebSinha2016c	[34]	N-L-N-O	1	1	4	0	1.12	0.06	[34]
LuDebSinha2016d	[34]	L-N-L-N	2	2	11	3	×	×	
LuDebSinha2016e	[34]	N-L-L-N	1	2	6	3	×	×	
LuDebSinha2016f	[34]	L-N-N-O	2	1	9	0	×	×	
MacalHurter1997	[36]	N-O-N-O	1	1	0	0	81.33	-0.33	[36]
Mirrlees1999	[38]	N-O-N-O	1	1	0	0	1	0.06	[38]
MitsosBarton2006Ex38	[39]	N-L-N-L	1	1	4	2	0	0	[39]
MitsosBarton2006Ex39	[39]	L-L-N-L	1	1	3	2	-1	-1	[39]

MitsosBarton2006Ex310	[39]	L-L-N-L	1	1	2	2	0.5	-0.1	[39]
MitsosBarton2006Ex311	[39]	L-L-N-L	1	1	2	2	-0.8	0	[39]
MitsosBarton2006Ex312	[39]	N-L-N-L	1	1	2	2	0	0	[39]
MitsosBarton2006Ex313	[39]	L-L-N-L	1	1	2	2	-1	0	[39]
MitsosBarton2006Ex314	[39]	N-L-N-L	1	1	2	2	0.25	-0.08	[39]
MitsosBarton2006Ex315	[39]	L-L-N-L	1	1	2	2	0	-0.83	[39]
MitsosBarton2006Ex316	[39]	L-L-N-L	1	1	2	2	-2	0	[39]
MitsosBarton2006Ex317	[39]	N-L-N-L	1	1	2	2	0.19	-0.02	[39]
MitsosBarton2006Ex318	[39]	N-L-N-L	1	1	2	2	-0.25	0	[39]
MitsosBarton2006Ex319	[39]	N-L-N-L	1	1	2	2	-0.26	0	[39]
MitsosBarton2006Ex320	[39]	N-L-N-L	1	1	2	2	0.31	-0.08	[39]
MitsosBarton2006Ex321	[39]	N-L-N-L	1	1	2	2	0.21	-0.07	[39]
MitsosBarton2006Ex322	[39]	N-L-N-N	1	1	2	3	0.21	-0.07	[39]
MitsosBarton2006Ex323	[39]	N-N-L-N	1	1	3	3	0.18	-1	[39]
MitsosBarton2006Ex324	[39]	N-L-N-L	1	1	2	2	-1.75	0	[39]
MitsosBarton2006Ex325	[39]	N-N-N-N	2	3	6	9	-1	-2	[39]
MitsosBarton2006Ex326	[39]	N-N-N-L	2	3	7	6	-2.35	-2	[39]
MitsosBarton2006Ex327	[39]	N-N-N-N	5	5	13	13	2	-1.1	[39]
MitsosBarton2006Ex328	[39]	N-N-N-N	5	5	13	13	-10	-3.1	[39]
MorganPatrone2006a	[40]	L-L-N-L	1	1	2	2	-1	0	[40]
MorganPatrone2006b	[40]	L-O-N-L	1	1	0	4	-1.25	0	[40]
MorganPatrone2006c	[40]	L-O-N-L	1	1	0	4	-1	-0.25	[40]
MuuQuy2003Ex1	[41]	N-L-N-L	1	2	2	3	-2.08	-0.59	[41]
MuuQuy2003Ex2	[41]	N-L-N-L	2	3	3	4	0.64	1.67	[41]
NieEtal2017Ex34	[42]	L-L-N-N	1	2	2	2	2	0	[42]
NieEtal2017Ex52	[42]	N-N-N-N	2	3	5	2	-1.71	-2.23	[42]
NieEtal2017Ex54	[42]	N-N-N-N	4	4	3	2	-0.44	-1.19	[42]
NieEtal2017Ex57	[42]	N-N-N-N	2	3	5	2	-2	-1	[42]
NieEtal2017Ex58	[42]	N-N-N-N	4	4	3	2	-3.49	-0.86	[42]
NieEtal2017Ex61	[42]	N-N-N-N	2	2	5	1	-1.02	-1.08	[42]
Outrata1990Ex1a	[43]	N-O-N-L	2	2	0	4	-8.92	-6.05	[43]
Outrata1990Ex1b	[43]	N-O-N-L	2	2	0	4	-7.56	-0.58	[43]
Outrata1990Ex1c	[43]	N-O-N-L	2	2	0	4	-12	-112.71	[43]
Outrata1990Ex1d	[43]	N-O-N-L	2	2	0	4	-3.6	-2	[43]
Outrata1990Ex1e	[43]	N-O-N-L	2	2	0	4	-3.15	-16.29	[43]
Outrata1990Ex2a	[43]	N-L-N-L	1	2	1	4	0.5	-14.53	[43]
Outrata1990Ex2b	[43]	N-L-N-L	1	2	1	4	0.5	-4.5	[43]
Outrata1990Ex2c	[43]	N-L-N-L	1	2	1	4	1.86	-10.93	[43]
Outrata1990Ex2d	[43]	N-L-N-N	1	2	1	4	0.92	-19.47	[43]
Outrata1990Ex2e	[43]	N-L-N-N	1	2	1	4	0.90	-14.94	[43]
Outrata1993Ex31	[44]	N-L-N-N	1	2	1	4	1.56	-11.67	[44]
Outrata1993Ex32	[44]	N-L-N-N	1	2	1	4	3.21	-20.53	[44]
Outrata1994Ex31	[45]	N-L-N-N	1	2	2	4	3.21	-20.53	[45]
OutrataCervinka2009	[46]	L-L-N-L	2	2	1	3	0	0	[46]
PaulaviciusEtal2017a	[47]	N-L-N-L	1	1	4	2	0.25	0	[47]
PaulaviciusEtal2017b	[47]	L-L-N-L	1	1	4	2	-2	-1.5	[47]
SahinCiric1998Ex2	[48]	N-L-N-L	1	1	2	3	5	4	[48]
ShimizuAiyoshi1981Ex1	[49]	N-L-N-L	1	1	3	3	100	0	[49]
ShimizuAiyoshi1981Ex2	[49]	N-L-N-L	2	2	3	4	225	100	[49]

ShimizuEtal1997a	[50]	N-O-N-L	1	1	0	3	×	×	
ShimizuEtal1997b	[50]	N-L-N-L	1	1	2	2	2250	197.75	[50]
SinhaMaloDeb2014TP3	[51]	N-N-N-N	2	2	3	4	-18.68	-1.02	[51]
SinhaMaloDeb2014TP6	[51]	N-L-N-L	1	2	1	6	-1.21	7.62	[51]
SinhaMaloDeb2014TP7	[51]	N-N-N-L	2	2	4	4	-1.96	1.96	[51]
SinhaMaloDeb2014TP8	[51]	N-L-N-L	2	2	5	6	0	100	[51]
SinhaMaloDeb2014TP9	[51]	N-O-N-L	10	10	0	20	0	1	[51]
SinhaMaloDeb2014TP10	[51]	N-O-N-L	10	10	0	20	0	1	[51]
TuyEtal2007	[52]	N-L-L-L	1	1	2	3	22.5	-1.52	[52]
Vogel2002	[53]	N-L-N-L	1	1	2	1	1	-2	[53]
WanWangLv2011	[54]	N-O-L-L	2	3	0	8	10.63	-0.5	[54]
YeZhu2010Ex42	[55]	N-L-N-L	1	1	2	1	1	-2	[55]
YeZhu2010Ex43	[55]	N-L-N-L	1	1	2	1	1.25	-2	[55]
Yezza1996Ex31	[56]	N-L-N-L	1	1	2	2	1.5	-2.5	[56]
Yezza1996Ex41	[56]	N-O-N-L	1	2	0	2	0.5	2.5	[56]
Zlobec2001a	[57]	N-O-L-L	1	2	0	3	-1	-1	[57]
Zlobec2001b	[57]	L-L-L-N	1	1	2	4	no	solution	[57]
DesignCentringP1	[71]	N-N-N-N	3	6	3	3	×	×	
DesignCentringP2	[71]	N-N-N-N	4	6	5	3	×	×	
DesignCentringP3	[71]	N-N-N-N	6	6	3	3	×	×	
DesignCentringP4	[71]	N-N-N-N	4	6	3	12	×	×	
NetworkDesignP1	[11]	N-L-N-L	5	5	5	6	300.5	419.8	[8]
NetworkDesignP2	[11]	N-L-N-L	5	5	5	6	142.9	81.95	[8]
OptimalControl	[37]	N-N-N-L	2	n_y	3	$2n_y$	×	×	
RobustPortfolioP1	[71]	L-N-N-N	N+1	N	N+3	N+1	1.15	0	[71]
RobustPortfolioP2	[71]	L-N-N-N	N+1	N	N+3	N+1	1.15	0	[71]
TollSettingP1	[11]	N-L-N-L	3	8	3	18	-7	12	[8]
TollSettingP2	[11]	N-L-N-L	3	18	3	38	-4.5	32	[8]
TollSettingP3	[11]	N-L-N-L	3	18	3	38	- 3.5	32	[8]
TollSettingP4	[11]	N-O-N-L	2	4	0	8	-4	14	[8]
TollSettingP5	[11]	N-O-N-L	1	4	0	8	-2.5	14	[8]
Linear bilevel examples									
AnandalinghamWhite1990	[58]	L-L-L-L	1	1	1	6	-49	15	[58]
Bard1984a	[75]	L-L-L-L	1	1	1	5	28/9	-60/9	[75]
Bard1984b	[75]	L-L-L-L	1	1	1	5	-37.6	1.6	[75]
Bard1991Ex2	[5]	L-L-L-L	1	2	1	5	-1	-1	[5]
BardFalk1982Ex2	[59]	L-L-L-L	2	2	2	5	-3.25	-4	[59]
Ben-AyedBlair1990a	[76]	L-L-L-L	1	2	2	4	-2.5	-5	[76]
Ben-AyedBlair1990b	[76]	L-L-L-L	1	1	1	4	-6	5	[76]
BialasKarwan1984a	[77]	L-L-L-L	1	2	1	7	-2	-0.5	[77]
BialasKarwan1984b	[77]	L-L-L-L	1	1	1	6	-11	11	[77]
CandlerTownsley1982	[60]	L-L-L-L	2	3	2	6	-29.2	3.2	[60]
ClarkWesterberg1988	[61]	L-O-L-L	1	1	0	3	-37	14	[61]
ClarkWesterberg1990b	[10]	L-L-L-L	1	2	2	5	-13	-4	[10]
GlackinEtal2009	[62]	L-L-L-L	2	1	3	3	6	0	[62]
HaurieSavardWhite1990	[78]	L-O-L-L	1	1	0	4	27	-3	[78]
HuHuangZhang2009	[63]	L-L-L-L	1	2	1	5	-76/9	-41/9	[63]
LanWenShihLee2007	[64]	L-L-L-L	1	1	1	7	-85.09	50.17	[64]
LiuHart1994	[67]	L-L-L-L	1	1	1	4	-16	4	[67]

`MershaDempe2006Ex1`	[65]	`L-L-L-L`	1	1	1	5	×	×	[65]
`MershaDempe2006Ex2`	[65]	`L-L-L-L`	1	1	2	2	-20	-6	[65]
`TuyEtal1993`	[79]	`L-L-L-L`	2	2	3	4	-3.25	-6	[79]
`TuyEtal1994`	[80]	`L-L-L-L`	2	2	3	3	6	0	[80]
`TuyEtal2007Ex3`	[52]	`L-L-L-L`	10	6	12	13	-467.46	-11.62	[52]
`VisweswaranEtal1996`	[68]	`L-L-L-L`	1	1	1	5	28/9	-60/9	[68]
`WangJiaoLi2005`	[66]	`L-L-L-L`	1	2	2	2	-1000	-1	[66]
Simple bilevel examples									
`FrankeEtal2018Ex53`	[73]	`N-L-N-L`	0	2	4	4	1	1	[73]
`FrankeEtal2018Ex511`	[73]	`N-O-L-L`	0	3	0	4	3	0	[73]
`FrankeEtal2018Ex513`	[73]	`L-O-L-N`	0	3	0	3	-1	0	[73]
`FrankeEtal2018Ex521`	[73]	`L-O-L-N`	0	2	0	3	-1	0	[73]
`MitsosBartonEx31`	[39]	`L-L-L-L`	0	1	2	2	1	-1	[39]
`MitsosBartonEx32`	[39]	`L-L-L-L`	0	1	3	2	no	solution	[39]
`MitsosBartonEx33`	[39]	`L-L-N-N`	0	1	2	3	-1	1	[39]
`MitsosBartonEx34`	[39]	`L-L-N-L`	0	1	2	2	1	-1	[39]
`MitsosBartonEx35`	[39]	`L-L-N-L`	0	1	2	2	0.5	-1	[39]
`MitsosBartonEx36`	[39]	`L-L-N-L`	0	1	2	2	-1	-1	[39]
`ShehuEtal2019Ex42`	[72]	`N-O-N-O`	0	n_y	0	0	×	×	

It is worth mentioning that some examples contain equalities constraints $H(x, y) = 0$ or $h(x, y) = 0$. As we mentioned in the beginning of this manuscript, equalities constraints can be transformed to inequalities ones. However, for the sake of clarity, we list those examples whose constraints contain equalities below.

TABLE 5. List of bilevel programs with equalities constraints.

Example name	**Ref.**	`F-G-H-f-g-h`	n_x	n_y	n_G	n_H	n_g	n_h
`DempeDutta2012Ex31`	[14]	`L-L-N-N-N-O`	2	2	2	1	2	0
`DempeFranke2011Ex41`	[16]	`N-L-L-N-L-O`	2	2	2	1	4	0
`DempeFranke2011Ex42`	[16]	`N-L-L-N-L-O`	2	2	2	1	3	0
`Zlobec2001b`	[57]	`L-L-O-L-L-N`	1	1	2	0	2	1
`NetworkDesignP1`	[11]	`N-L-O-N-O-L`	5	5	5	0	0	3
`NetworkDesignP2`	[11]	`N-L-O-N-O-L`	5	5	5	0	0	3
`OptimalControl`	[37]	`N-N-O-N-L-L`	2	n_y	3	0	n_y	$\frac{1}{2}n_y$
`RobustPortfolioP1`	[71]	`L-N-L-N-N-O`	N+1	N	N+1	1	N+1	0
`RobustPortfolioP2`	[71]	`L-N-L-N-N-O`	N+1	N	N+1	1	N+1	0
`TollSettingP1`	[11]	`N-L-O-N-L-L`	3	8	3	0	8	5
`TollSettingP2`	[11]	`N-L-O-N-L-L`	3	18	3	0	18	10
`TollSettingP3`	[11]	`N-L-O-N-L-L`	3	18	3	0	18	10
`TollSettingP4`	[11]	`N-O-O-N-L-L`	2	4	0	0	2	4
`TollSettingP5`	[11]	`N-O-O-N-L-L`	1	4	0	0	2	4

3. FORMULAS OF THE PROBLEMS

Here we provide formulas of the functions F, G, f, and g involved in problem (1.2) for all examples presented in this paper, together with true or best known values of the solutions, and some useful background information in some cases.

3.1. Nonlinear bilevel examples.

Problem name: `AiyoshiShimizu1984Ex2`

Source: [1]

Description: AiyoshiShimizu1984Ex2 is defined as follows

$$
\begin{aligned}
F(x,y) &:= 2x_1 + 2x_2 - 3y_1 - 3y_2 - 60\\
G(x,y) &:= \begin{bmatrix} x_1 + x_2 + y_1 - 2y_2 - 40 \\ x - 50_2 \\ -x \end{bmatrix}\\
f(x,y) &:= (y_1 - x_1 + 20)^2 + (y_2 - x_2 + 20)^2\\
g(x,y) &:= \begin{bmatrix} 2y - x + 10_2 \\ -y - 10_2 \\ y - 20_2 \end{bmatrix}
\end{aligned}
$$

Comment: Here, we write $50_2 := (50, 50)^\top$. Similar rule is applied into $10_2, 20_2$ and the context in the whole manuscript. The global optimal solution of the problem is $(25, 30, 5, 10)$ according to [1]. A local optimal one is $(0, 0, -10, -10)$ by [28].

Problem name: `AllendeStill2013`

Source: [2]

Description: AllendeStill2013 is defined as follows

$$
\begin{aligned}
F(x,y) &:= x_1^2 - 2x_1 + x_2^2 - 2x_2 + y_1^2 + y_2^2\\
G(x,y) &:= \begin{bmatrix} -x \\ -y \\ x_1 - 2 \end{bmatrix}\\
f(x,y) &:= y_1^2 - 2x_1y_1 + y_2^2 - 2x_2y_2\\
g(x,y) &:= \begin{bmatrix} (y_1 - 1)^2 - 0.25 \\ (y_2 - 1)^2 - 0.25 \end{bmatrix}
\end{aligned}
$$

Comment: The global optimal solution from [2] is $(0.5, 0.5, 0.5, 0.5)$.

Problem name: `AnEtal2009`

Source: [3]

Description: AnEtal2009 is defined as follows

$$
\begin{aligned}
F(x,y) &:= \tfrac{1}{2}(x^\top, y^\top)H(x^\top, y^\top)^\top + c_1^\top x + c_2^\top y\\
G(x,y) &:= \begin{bmatrix} -x \\ -y \\ Ax + By + d \end{bmatrix}\\
f(x,y) &:= y^\top Px + \tfrac{1}{2}y^\top Qy + q^\top y\\
g(x,y) &:= Dx + Ey + b
\end{aligned}
$$

with H, c_1, c_2, A, B, d, P, Q, q, D, E, and b, respectively defined as follows

$$H := \begin{bmatrix} -3.8 & 4.4 & 1.2 & -2.2 \\ 4.4 & -2.2 & 0.6 & 1.8 \\ 1.2 & 0.6 & 0.0 & 0.4 \\ -2.2 & 1.8 & 0.4 & 0.0 \end{bmatrix}, \quad c_1 := \begin{bmatrix} 935.74474 \\ 87.53654 \end{bmatrix}, \quad c_2 := \begin{bmatrix} 121.96196 \\ 299.24825 \end{bmatrix}$$

$$A := \begin{bmatrix} 0.00000 & 3.88889 \\ -2.00000 & 8.77778 \end{bmatrix}, \quad B := \begin{bmatrix} 4.88889 & 7.44444 \\ -5.11111 & 0.88889 \end{bmatrix}, \quad d := \begin{bmatrix} -61.57778 \\ -0.80000 \end{bmatrix}$$

$$P := \begin{bmatrix} -17.85000 & 6.57500 \\ 30.32500 & 30.32500 \end{bmatrix}, \quad Q := \begin{bmatrix} 21.10204 & 11.81633 \\ -5.11111 & -14.44898 \end{bmatrix}, \quad q := \begin{bmatrix} -18.21053 \\ 13.05263 \end{bmatrix}$$

$$D := \begin{bmatrix} 5.00000 & 7.44444 \\ -8.33333 & 3.00000 \\ -8.66667 & -8.55556 \\ 6.44444 & -5.11111 \end{bmatrix}, \quad E := \begin{bmatrix} 3.88889 & 1.77778 \\ 6.88889 & 6.11111 \\ -5.33333 & -7.00000 \\ 1.44444 & 4.44444 \end{bmatrix}, \quad b := \begin{bmatrix} -39.62222 \\ -60.00000 \\ 72.37778 \\ -17.28889 \end{bmatrix}$$

Comment: $(0.200001, 1.999997, 3.999998, 4.600005)$ is the global optimal solution of the problem; cf. [3].

Problem name: `Bard1988Ex1`

Source: [4]

Description: Bard1988Ex1 is defined as follows

$$\begin{aligned} F(x,y) &:= (x-5)^2 + (2y+1)^2 \\ G(x,y) &:= -x \\ f(x,y) &:= (y-1)^2 - 1.5xy \\ g(x,y) &:= \begin{bmatrix} -3x + y + 3 \\ x - 0.5y - 4 \\ x + y - 7 \\ -y \end{bmatrix} \end{aligned}$$

Comment: $(1, 0)$ is the global optimum and $(5, 2)$ is a local optimal point.

Problem name: `Bard1988Ex2`

Source: [4]

Description: Bard1988Ex2 is defined as follows

$$\begin{aligned} F(x,y) &:= (200 - y_1 - y_3)(y_1 + y_3) + (160 - y_2 - y_4)(y_2 + y_4) \\ G(x,y) &:= \begin{bmatrix} x_1 + x_2 + x_3 + x_4 - 40 \\ -[10, 5, 15, 20]^\top + x \\ -x \end{bmatrix} \\ f(x,y) &:= (y_1 - 4)^2 + (y_2 - 13)^2 + (y_3 - 35)^2 + (y_4 - 2)^2 \\ g(x,y) &:= \begin{bmatrix} 0.4y_1 + 0.7y_2 - x_1 \\ 0.6y_1 + 0.3y_2 - x_2 \\ 0.4y_3 + 0.7y_4 - x_3 \\ 0.6y_3 + 0.3y_4 - x_4 \\ -[20, 20, 40, 40]^\top + y \\ -y \end{bmatrix} \end{aligned}$$

Comment: This version of the problem is taken from [11]. The original one in [4] has two lower-level problem. The upper-and lower-level optimal value are respectively obtained as -6600.00 and 57.48 in the former paper. However, the global upper-and lower-level optimal value should be -6600.00 and 54 according to [4].

Problem name: `Bard1988Ex3`

Source: [4]

Description: `Bard1988Ex3` is defined as follows

$$\begin{array}{rcl} F(x,y) & := & -x_1^2 - 3x_2 - 4y_1 + y_2^2 \\ G(x,y) & := & \begin{bmatrix} x_1^2 + 2x_2 - 4 \\ -x \end{bmatrix} \\ f(x,y) & := & 2x_1^2 + y_1^2 - 5y_2 \\ g(x,y) & := & \begin{bmatrix} -x_1^2 + 2x_1 - x_2^2 + 2y_1 - y_2 - 3 \\ -x_2 - 3y_1 + 4y_2 + 4 \\ -y \end{bmatrix} \end{array}$$

Comment: The global upper-and lower-level optimal objective value are respectively obtained as -12.68 and -1.02 in the paper [8].

Problem name: `Bard1991Ex1`

Source: [5]

Description: Bard1991Ex1 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x + y_2 \\ G(x,y) & := & \begin{bmatrix} -x + 2 \\ x - 4 \end{bmatrix} \\ f(x,y) & := & 2y_1 + xy_2 \\ g(x,y) & := & \begin{bmatrix} x - y_1 - y_2 + 4 \\ -y \end{bmatrix} \end{array}$$

Comment: The global optimal solution of the problem is $(2, 6, 0)$; cf. [5].

Problem name: `BardBook1998`

Source: [6]

Description: BardBook1998 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & (y_1 - x_1 + 20)^2 + (y_2 - x_2 + 20)^2 \\ G(x,y) & := & \begin{bmatrix} x - 50_2 \\ -x \end{bmatrix} \\ f(x,y) & := & 2x_1 + 2x_2 - 3y_1 - 3y_2 - 60 \\ g(x,y) & := & \begin{bmatrix} x_1 + x_2 + y_1 - 2y_2 - 40 \\ 2y - x + 10_2 \\ y - 20_2 \\ -y - 10_2 \end{bmatrix} \end{array}$$

Comment: The global optimal solution is $(25\ 30\ 5\ 10)$.

Problem name: `CalamaiVicente1994a`

Source: [7]

Description: CalamaiVicente1994a is defined as follows

$$\begin{array}{rcl} F(x,y) & := & \frac{1}{2}(x-1)^2 + \frac{1}{2}y^2 \\ f(x,y) & := & \frac{1}{2}y - xy \\ g(x,y) & := & \begin{bmatrix} x-y-1 \\ -x-y+1 \\ x+y-\rho \end{bmatrix} \end{array}$$

Comment: It is assumed in [7] that the parameter $\rho \geq 1$. We consider the following scenarios studied in the latter reference:

(i) For $\rho = 1$, the point $(1, 0)$ is global optimum of the problem.
(ii) For $1 < \rho < 2$, the point $\frac{1}{2}(1+\rho, -1+\rho)$ is a global optimal solution, while $\frac{1}{2}(1, 1)$ is a local optimal solution of the problem.
(iii) For $\rho = 2$, the points $\frac{1}{2}(1, 1)$ and $\frac{1}{2}(3, 1)$ are global optimal solution.
(iv) For $\rho > 2$, the point $\frac{1}{2}(1, 1)$ is global optimum of the problem.

Problem name: `CalamaiVicente1994b`
Source: [7]
Description: CalamaiVicente1994b is defined as follows

$$\begin{array}{rcl} F(x,y) & := & \frac{1}{2}\sum_{i=1}^{4}(x_i-1)^2 + \sum_{i=1}^{2} y_i^2 \\ f(x,y) & := & \sum_{i=1}^{2}\left(\frac{1}{2}y_i^2 - x_i y_i\right) \\ g(x,y) & := & \begin{bmatrix} x-y-1_2 \\ -x-y+1_2 \\ x_1+y_1-1.5 \\ x_1+y_2-3 \end{bmatrix} \end{array}$$

Comment: The global optimal solution is $(1.25\ 0.5\ 1\ 1\ 0.25\ 0.5)$ with upper-and lower-level optimal objective value being 0.3125 and -0.4063.

Problem name: `CalamaiVicente1994c`
Source: [7]
Description: CalamaiVicente1994c is defined as follows

$$\begin{array}{rcl} F(x,y) & := & \frac{1}{2}x^\top Ax + \frac{1}{2}y^\top By + a^\top x + 2 \\ f(x,y) & := & \frac{1}{2}y^\top By + x^\top Cy \\ g(x,y) & := & Dx + Ey + d \end{array}$$

with A, B, a, B and C respectively as follows

$$A := \begin{bmatrix} 197.2 & 32.4 & -129.6 & -43.2 \\ 32.4 & 110.8 & -43.2 & -14.4 \\ -129.6 & -43.2 & 302.8 & -32.4 \\ -43.2 & -14.4 & -32.4 & 289.2 \end{bmatrix}, \quad B := \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix}, \quad a := \begin{bmatrix} -8.56 \\ -9.52 \\ -9.92 \\ -16.64 \end{bmatrix}$$

$$C := \begin{bmatrix} -132.4 & -10.8 \\ -10.8 & -103.6 \\ 43.2 & 14.4 \\ 14.4 & 4.8 \end{bmatrix}, \quad D := \begin{bmatrix} 13.24 & 1.08 & -4.32 & -1.44 \\ 1.08 & 10.36 & -1.44 & -0.48 \\ 13.24 & 1.08 & -4.32 & -1.44x_4 \\ 1.08 & 10.36 & -1.44 & -0.48 \\ -13.24 & -1.08 & +4.32 & +1.44 \\ -1.08 & -10.36 & +1.44 & +0.48 \end{bmatrix}$$

$$E := \begin{bmatrix} -10 & 0 \\ 0 & -10 \\ 10 & 0 \\ 0 & 10 \\ -10 & 0 \\ 0 & -10 \end{bmatrix}, \quad d := \begin{bmatrix} -1 \\ -1 \\ -1.5 \\ -3 \\ 1 \\ 1 \end{bmatrix}$$

Comment: According to [7], the problem has a unique global optimal solution

$$(0.13085,\ 0.05195,\ 0.1022,\ 0.0674,\ 0.025,\ 0.05)$$

with the corresponding upper-level and lower-level objective function values being 0.3125 and -0.4063..

Problem name: `CalveteGale1999P1`

Source: [9]

Description: CalveteGale1999P1 is defined as follows

$$\begin{aligned} F(x,y) &:= -8x_1 - 4x_2 + y_1 - 40y_2 - 4y_3 \\ G(x,y) &:= -x \\ f(x,y) &:= \frac{1 + x_1 + x_2 + 2y_1 - y_2 + y_3}{6 + 2x_1 + y_1 + y_2 - 3y_3} \\ g(x,y) &:= \begin{bmatrix} -y \\ -y_1 + y_2 + y_3 - 1 \\ 2x_1 - y_1 + 2y_2 - \frac{1}{2}y_3 - 1 \\ 2x_2 + 2y_1 - y_2 - \frac{1}{2}y_3 - 1 \end{bmatrix} \end{aligned}$$

Comment: The global optimal value of the upper-level objective function is -29.2 and can be achieved at $(0.0, 0.9, 0.0, 0.6, 0.4)$, for example; cf. [23].

Problem name: `ClarkWesterberg1990a`

Source: [10]

Description: ClarkWesterberg1990a is defined as follows

$$\begin{aligned} F(x,y) &:= (x-3)^2 + (y-2)^2 \\ G(x,y) &:= \begin{bmatrix} x-8 \\ -x \end{bmatrix} \\ f(x,y) &:= (y-5)^2 \\ g(x,y) &:= \begin{bmatrix} -2x+y-1 \\ x-2y+2 \\ x+2y-14 \end{bmatrix} \end{aligned}$$

Comment: The global optimal solution of the problem is $(1.0, 3.0)$; cf. [47].

Problem name: `Colson2002BIPA1`

Source: [11]

Description: Colson2002BIPA1 is defined as follows

$$\begin{aligned} F(x,y) &:= (10-x)^3 + (10-y)^3 \\ G(x,y) &:= \begin{bmatrix} x-5 \\ -x+y \\ -x \end{bmatrix} \\ f(x,y) &:= (x+2y-15)^4 \\ g(x,y) &:= \begin{bmatrix} x+y-20 \\ y-20 \\ -y \end{bmatrix} \end{aligned}$$

Comment: A global optimal solution is $(5, 5)$.

Problem name: `Colson2002BIPA2`

Source: [11]

Description: Colson2002BIPA2 is defined as follows

$$\begin{aligned} F(x,y) &:= (x-5)^2 + (2y+1)^2 \\ G(x,y) &:= -x \\ f(x,y) &:= (y-1)^2 - 1.5xy + x^3 \\ g(x,y) &:= \begin{bmatrix} -3x+y+3 \\ x-0.5y-4 \\ x+y-7 \\ -y \end{bmatrix} \end{aligned}$$

Comment: The best known solution is $(1, 0)$; cf. [8].

Problem name: `Colson2002BIPA3`

Source: [11]

Description: Colson2002BIPA3 is defined as follows

$$\begin{aligned} F(x,y) &:= (x-5)^4 + (2y+1)^4 \\ G(x,y) &:= \begin{bmatrix} x+y-4 \\ -x \end{bmatrix} \\ f(x,y) &:= \exp(-x+y) + x^2 + 2xy + y^2 + 2x + 6y \\ g(x,y) &:= \begin{bmatrix} -x+y-2 \\ -y \end{bmatrix} \end{aligned}$$

Comment: The best known solution is $(4, 0)$; cf. [8].

Problem name: `Colson2002BIPA4`

Source: [11]

Description: Colson2002BIPA4 as defined as follows

$$\begin{aligned} F(x,y) &:= x^2 + (y-10)^2 \\ G(x,y) &:= \begin{bmatrix} x+2y-6 \\ -x \end{bmatrix} \\ f(x,y) &:= x^3 + 2y^3 + x - 2y - x^2 \\ g(x,y) &:= \begin{bmatrix} -x+2y-3 \\ -y \end{bmatrix} \end{aligned}$$

Comment: The best known solution is $(0, 0.6039)$; cf. [8].

Problem name: `Colson2002BIPA5`

Source: [11]

Description: Colson2002BIPA5 is defined as follows

$$\begin{aligned} F(x,y) &:= (x-y_2)^4 + (y_1-1)^2 + (y_1-y_2)^2 \\ G(x,y) &:= -x \\ f(x,y) &:= 2x + \exp y_1 + y_1^2 + 4y_1 + 2y_2^2 - 6y_2 \\ g(x,y) &:= \begin{bmatrix} 6x + y_1^2 + \exp y_2 - 15 \\ 5x + y_1^4 - y_2 - 25 \\ -[4,2]^\top + y \\ -y \end{bmatrix} \end{aligned}$$

Comment: The best known solution is $(1.94, 0, 1.21)$; cf. [8].

Problem name: `Dempe1992a`

Source: [12]

Description: Dempe1992a is defined as follows

$$\begin{aligned} F(x,y) &:= y_2 \\ G(x,y) &:= x_1^2 + (x_2+1)^2 - 1 \\ f(x,y) &:= \tfrac{1}{2}(y_1-1)^2 + \tfrac{1}{2}y_2^2 \\ g(x,y) &:= \begin{bmatrix} y_1 + y_2 x_1 + x_2 \\ y_1 \end{bmatrix} \end{aligned}$$

Comment: One possible solution is $(0, 0, 0, -0.5)$.

Problem name: `Dempe1992b`

Source: [12]

Description: Dempe1992b is defined as follows

$$\begin{aligned} F(x,y) &:= (x-3.5)^2 + (y+4)^2 \\ f(x,y) &:= (y-3)^2 \\ g(x,y) &:= y^2 - x \end{aligned}$$

Comment: The global upper-and lower-level optimal values are respectively obtained as 31.25 and 4.00 in the paper [8].

Problem name: `DempeDutta2012Ex24`

Source: [14]

Description: DempeDutta2012Ex24 is defined as follows

$$\begin{aligned} F(x,y) &:= (x-1)^2 + y^2 \\ f(x,y) &:= x^2 y \\ g(x,y) &:= y^2 \end{aligned}$$

Comment: The global optimal solution of the problem is $(1,0)$; cf. [14].

Problem name: `DempeDutta2012Ex31`

Source: [14]

Description: DempeDutta2012Ex31 is defined as follows

$$\begin{aligned} F(x,y) &:= -y_2 \\ G(x,y) &:= \begin{bmatrix} -x \\ y_1 y_2 \\ -y_1 y_2 \end{bmatrix} \\ f(x,y) &:= y_1^2 + (y_2+1)^2 \\ g(x,y) &:= \begin{bmatrix} (y_1-x_1)^2 + (y_2-x_1-1)^2 - 1 \\ (y_1+x_2)^2 + (y_2-x_2-1)^2 - 1 \end{bmatrix} \end{aligned}$$

Comment: The point $(0.71, 0.71, 0, 1)$ is global optimal solution of the problem provided in [14, 42].

Problem name: `DempeEtal2012`

Source: [15]

Description: DempeEtal2012 is defined as follows

$$\begin{aligned} F(x,y) &:= x \\ G(x,y) &:= \begin{bmatrix} -1-x \\ x-1 \end{bmatrix} \\ f(x,y) &:= xy \\ g(x,y) &:= \begin{bmatrix} -y \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: The global optimal solution is $(-1,1)$; cf. [15].

Problem name: `DempeFranke2011Ex41`

Source: [16]

Description: DempeFranke2011Ex41 is defined as follows

$$\begin{aligned} F(x,y) &:= x_1 + y_1^2 + y_2^2 \\ G(x,y) &:= \begin{bmatrix} -1-x_1 \\ -1+x_1 \\ -1-x_2 \\ 1+x_2 \end{bmatrix} \\ f(x,y) &:= x^\top y \\ g(x,y) &:= \begin{bmatrix} -2y_1+y_2 \\ y_1-2 \\ -y_2 \\ y_2-2 \end{bmatrix} \end{aligned}$$

Comment: The global optimal solution is $(0,-1,1,2)$; cf. [16].

Problem name: `DempeFranke2011Ex42`

Source: [16]

Description: DempeFranke2011Ex42 is defined as follows

$$
\begin{aligned}
F(x,y) &:= x_1 + (y_1-1)^2 + y_2^2 \\
G(x,y) &:= \begin{bmatrix} -1-x_1 \\ -1+x_1 \\ -1-x_2 \\ 1+x_2 \end{bmatrix} \\
f(x,y) &:= x^\top y \\
g(x,y) &:= \begin{bmatrix} -y_1+y_2-1 \\ y_1+y_2-3.5 \\ y_2-2 \end{bmatrix}
\end{aligned}
$$

Comment: The global optimal solution is $(1,-1,0,1)$; cf. [16].

Problem name: `DempeFranke2014Ex38`

Source: [17]

Description: DempeFranke2014Ex38 is defined as follows

$$
\begin{aligned}
F(x,y) &:= 2x_1 + x_2 + 2y_1 - y_2 \\
G(x,y) &:= \begin{bmatrix} -1-x_1 \\ -1+x_1 \\ -1-x_2 \\ x_2+0.75 \end{bmatrix} \\
f(x,y) &:= x^\top y \\
g(x,y) &:= \begin{bmatrix} -2y_1+y_2 \\ y_1-2 \\ -y_2 \\ y_2-2 \end{bmatrix}
\end{aligned}
$$

Comment: The global optimal solution is $(-1,-1,2,2)$; cf. [17].

Problem name: `DempeLohse2011Ex31a`

Source: [18]

Description: DempeLohse2011Ex31a is defined as follows

$$
\begin{aligned}
F(x,y) &:= (x_1-0.5)^2 + (x_2-0.5)^2 - 3y_1 - 3y_2 \\
f(x,y) &:= x_1y_1 + x_2y_2 \\
g(x,y) &:= \begin{bmatrix} y_1+y_2-2 \\ -y_1+y_2 \\ -y \end{bmatrix}
\end{aligned}
$$

Comment: Authors in [18] claimed the point $(0.5,0.5,1,1)$ is the unique global optimal solution to this problem, which actually is not correct since solution set of the lower level problem is $\{(0,0)\}$ when $x=(0.5,0.5)$; The true unique global solution should be $(0,0,1,1)$.

Problem name: `DempeLohse2011Ex31b`

Source: [18]

Description: DempeLohse2011Ex31b is defined as follows

$$\begin{array}{rcl} F(x,y) & := & (x_1-0.5)^2+(x_2-0.5)^2+x_3^2-3y_1-3y_2-6y_3 \\ f(x,y) & := & x_1y_1+x_2y_2+x_3y_3 \\ g(x,y) & := & \begin{bmatrix} y_1+y_2+y_3-2 \\ -y_1+y_2 \\ -y \end{bmatrix} \end{array}$$

Comment: The point $(0.5, 0.5, 0, 1, 1, 0)$ is a local optimal solution of the problem according to [18]. While a suggested global optimal solution is $(0.5, 0.5, 0, 0, 0, 2)$.

Problem name: `DeSilva1978`

Source: [19]

Description: DeSilva1978 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x_1^2-2x_1+x_2^2-2x_2+y_1^2+y_2^2 \\ f(x,y) & := & (y_1-x_1)^2+(y_2-x_2)^2 \\ g(x,y) & := & \begin{bmatrix} -y+(0.5)_2 \\ y-(1.5)_2 \end{bmatrix} \end{array}$$

Comment: The global optimal upper-and lower-level values obtained in [8] are -1.00 and 0.00, respectively. Corresponding global optimal solution is $(0.5, 0.5, 0.5, 0.5)$.

Problem name: `FalkLiu1995`

Source: [20]

Description: FalkLiu1995 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x_1^2-3x_1+x_2^2-3x_2+y_1^2+y_2^2 \\ f(x,y) & := & (y_1-x_1)^2+(y_2-x_2)^2 \\ g(x,y) & := & \begin{bmatrix} -y+(0.5)_2 \\ y-(1.5)_2 \end{bmatrix} \end{array}$$

Comment: The optimal solution is $(\sqrt{3}/2, \sqrt{3}/2, \sqrt{3}/2, \sqrt{3}/2)$ according to [20].

Problem name: `FloudasEtal2013`

Source: [21]

Description: FloudasEtal2013 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & 2x_1+2x_2-3y_1-3y_2-60 \\ G(x,y) & := & \begin{bmatrix} -x \\ x-50_2 \end{bmatrix} \\ f(x,y) & := & (y_1-x_1+20)^2+(y_2-x_2+20)^2 \\ g(x,y) & := & \begin{bmatrix} x_1+x_2+y_1-2y_2-40 \\ 2y-x+10_2 \\ -y-10_2 \\ y-20_2 \end{bmatrix} \end{array}$$

Comment: The global optimal solution of the problem is $(0, 0, -10, -10)$; [52].

Problem name: `FloudasZlobec1998`

Source: [22]

Description: FloudasZlobec1998 is defined as follows

$$
\begin{aligned}
F(x,y) &:= x^3 y_1 + y_2 \\
G(x,y) &:= \begin{bmatrix} x-1 \\ -x \end{bmatrix} \\
f(x,y) &:= -y_2 \\
g(x,y) &:= \begin{bmatrix} -y_1 - 1 \\ y_1 - 1 \\ -y_2 \\ y_2 - 100 \\ xy_1 - 10 \\ y_1^2 + xy_2 - 1 \end{bmatrix}
\end{aligned}
$$

Comment: Notice that explicit bounds on the variable y were added. This is same as [39]. The global optimal solution is $(1, 0, 1)$ according to [23, 39].

Problem name: `GumusFloudas2001Ex1`

Source: [23]

Description: GumusFloudas2001Ex1 is defined as follows

$$
\begin{aligned}
F(x,y) &:= 16x^2 + 9y^2 \\
G(x,y) &:= \begin{bmatrix} -x \\ x - 12.5 \\ -4x + y \end{bmatrix} \\
f(x,y) &:= (x + y - 20)^4 \\
g(x,y) &:= \begin{bmatrix} -y \\ y - 50 \\ 4x + y - 50 \end{bmatrix}
\end{aligned}
$$

Comment: The global optimal solution of the problem is $(11.25, 5)$; cf. [39].

Problem name: `GumusFloudas2001Ex3`

Source: [23]

Description: GumusFloudas2001Ex3 is defined as follows

$$
\begin{aligned}
F(x,y) &:= -8x_1 - 4x_2 + y_1 - 40y_2 - 4y_3 \\
G(x,y) &:= \begin{bmatrix} -x \\ x - 2_2 \end{bmatrix} \\
f(x,y) &:= \frac{1 + x_1 + x_2 + 2y_1 - y_2 + y_3}{6 + 2x_1 + y_1 + y_2 - 3y_3} \\
g(x,y) &:= \begin{bmatrix} -y \\ y - 2_3 \\ -y_1 + y_2 + y_3 - 1 \\ 2x_1 - y_1 + 2y_2 - \frac{1}{2}y_3 - 1 \\ 2x_2 + 2y_1 - y_2 - \frac{1}{2}y_3 - 1 \end{bmatrix}
\end{aligned}
$$

Comment: The global optimal solution of the problem is $(0, 0.9, 0, 0.6, 0.4)$; cf. [39].

Problem name: `GumusFloudas2001Ex4`

Source: [23]

Description: GumusFloudas2001Ex4 is defined as follows

$$
\begin{aligned}
F(x,y) &:= (x-3)^2+(y-2)^2\\
G(x,y) &:= \begin{bmatrix} -x \\ x-8 \\ -2x+y-1 \\ x-2y+2 \\ x+2y-14 \end{bmatrix}\\
f(x,y) &:= (y-5)^2\\
g(x,y) &:= \begin{bmatrix} -y \\ y-10 \end{bmatrix}
\end{aligned}
$$

Comment: The global optimal solution of the problem is $(3, 5)$; cf. [39].

Problem name: `GumusFloudas2001Ex5`

Source: [23]

Description: GumusFloudas2001Ex5 is defined as follows

$$
\begin{aligned}
F(x,y) &:= x\\
G(x,y) &:= \begin{bmatrix} -x+0.1 \\ x-10 \end{bmatrix}\\
f(x,y) &:= -y_1+0.5864y_1^{0.67}\\
g(x,y) &:= \begin{bmatrix} -y+(0.1)_2 \\ y-10_2 \\ \frac{0.0332333}{y_2}+0.1y_1-1 \\ 4\frac{x}{y_2}+2\frac{x^{-0.71}}{y_2}+0.0332333x^{-1.3}-1 \end{bmatrix}
\end{aligned}
$$

Comment: The global optimal solution is $(0.193616, 9.9667667, 10)$; cf. [39].

Problem name: `HatzEtal2013`

Source: [24]

Description: HatzEtal2013 is defined as follows

$$
\begin{aligned}
F(x,y) &:= -x+2y_1+y_2\\
f(x,y) &:= (x-y_1)^2+y_2^2\\
g(x,y) &:= -y
\end{aligned}
$$

Comment: The global optimal solution of the problem is $(0, 0, 0)$; cf. [24].

Problem name: `HendersonQuandt1958`

Source: [25]

Description: HendersonQuandt1958 is defined as follows

$$
\begin{aligned}
F(x,y) &:= \tfrac{1}{2}x^2+\tfrac{1}{2}xy-95x\\
G(x,y) &:= \begin{bmatrix} x-200 \\ -x \end{bmatrix}\\
f(x,y) &:= y^2+(\tfrac{1}{2}x-100)y\\
g(x,y) &:= -y
\end{aligned}
$$

Comment: The best known solution from [27] is $(93.33333, 26.667)$.

Problem name: `HenrionSurowiec2011`

Source: [26]

Description: HenrionSurowiec2011 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x^2 + cy \\ f(x,y) & := & 0.5y^2 - xy \end{array}$$

Comment: Here, c is a real-valued parameter. The global optimal solution of the problem is $-0.5c(1,1)$; cf. [26].

Problem name: `IshizukaAiyoshi1992a`

Source: [28]

Description: IshizukaAiyoshi1992a is defined as follows

$$\begin{array}{rcl} F(x,y) & := & xy_2^2 \\ G(x,y) & := & -x - M \\ f(x,y) & := & y_1 \\ g(x,y) & := & \begin{bmatrix} -x \\ -x - y_1 \\ y_1 - x \\ -M - y_1 - y_2 \\ y_1 + y_2 - M \end{bmatrix} \end{array}$$

Comment: Here, M is assumed to be an arbitrarily large number such that $M > 1$. From [28], $(x^*, -M, 0)$ is the global optimal solution where $x^* \in [0, M]$.

Problem name: `KleniatiAdjiman2014Ex3`

Source: [29]

Description: KleniatiAdjiman2014Ex3 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x - y \\ G(x,y) & := & \begin{bmatrix} -x - 1 \\ x - 1 \end{bmatrix} \\ f(x,y) & := & 0.5xy^2 - xy^3 \\ g(x,y) & := & \begin{bmatrix} -y - 1 \\ y - 1 \end{bmatrix} \end{array}$$

Comment: $(0,1)$ is the global optimal solution of the problem according to [29, 47].

Problem name: `KleniatiAdjiman2014Ex4`

Source: [29]

Description: KleniatiAdjiman2014Ex4 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & -\sum_{j=1}^{5}\left(x_j^2 + y_j^2\right) \\ G(x,y) & := & \begin{bmatrix} y_1y_2 - x_1 \\ x_2y_1^2 \\ x_1 - \exp x_2 + y_3 \\ -x - 1_5 \\ x - 1_5 \end{bmatrix} \\ f(x,y) & := & y_1^3 + y_2^2x_1 + y_2^2x_2 + 0.1y_3 + (y_4^2 + y_5^2)x_3x_4x_5 \\ g(x,y) & := & \begin{bmatrix} x_1 - y_3^2 - 0.2 \\ -y - 1_5 \\ y - 1_5 \end{bmatrix} \end{array}$$

Comment: $(1.0, -(1.0)_9)$ is the best known solution of the problem according to [29, 47].

Problem name: `LamparielloSagratella2017Ex23`

Source: [30]

Description: LamparielloSagratella2017Ex23 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x \\ G(x,y) & := & \begin{bmatrix} -x-1 \\ x-1 \end{bmatrix} \\ f(x,y) & := & (x-y_1)^2+(y_2+1)^2 \\ g(x,y) & := & \begin{bmatrix} y_1^3-y_2 \\ -y_2 \end{bmatrix} \end{array}$$

Comment: The global optimal solution is $(-1,-1,0)$; cf. [30].

Problem name: `LamparielloSagratella2017Ex31`

Source: [31]

Description: LamparielloSagratella2017Ex31 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x^2+y^2 \\ G(x,y) & := & -x+1 \\ f(x,y) & := & y \\ g(x,y) & := & -x-y+1 \end{array}$$

Comment: The global optimal solution is $(1,0)$; cf. [31].

Problem name: `LamparielloSagratella2017Ex32`

Source: [31]

Description: LamparielloSagratella2017Ex32 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x^2+y^2 \\ f(x,y) & := & (x+y-1)^2 \end{array}$$

Comment: The global optimal solution is $(0.5,0.5)$; cf. [31].

Problem name: `LamparielloSagratella2017Ex33`

Source: [31]

Description: LamparielloSagratella2017Ex33 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x^2+(y_1+y_2)^2 \\ G(x,y) & := & -x+0.5 \\ f(x,y) & := & y_1 \\ g(x,y) & := & \begin{bmatrix} -x-y_1-y_2+1 \\ -y \end{bmatrix} \end{array}$$

Comment: The global optimal solution is $(0.5,0,0.5)$; cf. [31].

Problem name: `LamparielloSagratella2017Ex35`

Source: [31]

Description: LamparielloSagratella2017Ex35 is defined as follows

$$\begin{aligned} F(x,y) &:= x^2+y^2 \\ G(x,y) &:= \begin{bmatrix} -1-x \\ x-1 \end{bmatrix} \\ f(x,y) &:= -y \\ g(x,y) &:= \begin{bmatrix} 2x+y-2 \\ -y \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: The global optimal solution is $(\frac{4}{5}, \frac{2}{5})$; cf. [31].

Problem name: `LucchettiEtal1987`

Source: [33]

Description: LucchettiEtal1987 is defined as follows

$$\begin{aligned} F(x,y) &:= 0.5(1-x)+xy \\ G(x,y) &:= \begin{bmatrix} -x \\ x-1 \end{bmatrix} \\ f(x,y) &:= (x-1)y \\ g(x,y) &:= \begin{bmatrix} -y \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: The global optimal solution of the problem is $(1,0)$; cf. [33].

Problem name: `LuDebSinha2016a`

Source: [34]

Description: LuDebSinha2016a is defined as follows

$$\begin{aligned} F(x,y) &:= 2-\exp\left[-\left(\tfrac{0.2y-x+0.6}{0.055}\right)^{0.4}\right]-0.8\exp\left[-\left(\tfrac{0.15y-0.4+x}{0.3}\right)^{2}\right] \\ G(x,y) &:= \begin{bmatrix} -x \\ x-1 \\ -y \\ y-2 \end{bmatrix} \\ f(x,y) &:= 2-\exp\left[-\left(\tfrac{1.5y-x}{0.055}\right)^{0.4}\right]-0.8\exp\left[-\left(\tfrac{2y-3+x}{0.5}\right)^{2}\right] \end{aligned}$$

Comment: $(0.2, 1.4)$ is the best known solution for the problem; [34].

Problem name: `LuDebSinha2016b`

Source: [34]

Description: LuDebSinha2016b is defined as follows

$$\begin{aligned} F(x,y) &:= (x-0.5)^2+(y-1)^2 \\ G(x,y) &:= \begin{bmatrix} -x \\ x-1 \\ -y \\ y-2 \end{bmatrix} \\ f(x,y) &:= 2-\exp\left[-\left(\tfrac{1.5y-x}{0.055}\right)^{0.4}\right]-0.8\exp\left[-\left(\tfrac{2y-3+x}{0.5}\right)^{2}\right] \end{aligned}$$

Comment: $(0.5, 1)$ is the best known solution for the problem; [34].

Problem name: `LuDebSinha2016c`

Source: [34]

Description: LuDebSinha2016c is defined as follows

$$
\begin{aligned}
F(x,y) &:= 2-\exp\left[-\left(\frac{0.2y-x+0.6}{0.055}\right)^{0.4}\right]-0.8\exp\left[-\left(\frac{0.15y-0.4+x}{0.3}\right)^{2}\right]\\
G(x,y) &:= \begin{bmatrix} -x \\ x-1 \\ -y \\ y-2 \end{bmatrix}\\
f(x,y) &:= (x-0.5)^2+(y-1)^2
\end{aligned}
$$

Comment: $(0.26, 1)$ is the best known possible solution for the problem; [34].

Problem name: `LuDebSinha2016d`

Source: [34]

Description: LuDebSinha2016d is defined as follows

$$
\begin{aligned}
F(x,y) &:= -x_2\\
g(x,y) &:= \begin{bmatrix} -\left(\frac{y_1}{14}+\frac{16}{7}\right)(x_1-2)^2+x_2 \\ -x_2+\left(\frac{y_1}{14}+\frac{16}{7}\right)(x_1-5) \\ -\left[x_1+4-\left(\frac{y_1}{14}+\frac{16}{7}\right)\right]\left[x_1+8-\left(\frac{y_1}{14}+\frac{16}{7}\right)\right]+x_2 \\ -4-x_1 \\ -10+x_1 \\ -100-x_2 \\ -200+x_2 \\ -4-y_1 \\ -10+y_1 \\ -100-y_2 \\ -200+y_2 \end{bmatrix}\\
f(x,y) &:= -y_2\\
g(x,y) &:= \begin{bmatrix} -\left(\frac{x_1}{14}+\frac{16}{7}\right)(y_1-2)^2+y_2 \\ -y_2+12.5\left(\frac{x_1}{14}+\frac{16}{7}\right)(y_1-5) \\ -5\left[y_1+4-\left(\frac{x_1}{14}+\frac{16}{7}\right)\right]\left[y_1+8-\left(\frac{x_1}{14}+\frac{16}{7}\right)\right]+y_2 \end{bmatrix}
\end{aligned}
$$

Comment: Solutions are unknown. A possible solution is $(10, 192, 10, 192)$.

Problem name: `LuDebSinha2016e`

Source: [34]

Description: LuDebSinha2016e is defined as follows

$$
\begin{aligned}
F(x,y) &:= \left(\tfrac{y_2-50}{30}\right)^2 + \left(\tfrac{x-2.5}{0.2}\right)^2 \\
G(x,y) &:= \begin{bmatrix} 2-x \\ -3+x \\ -4-y_1 \\ -10+y_1 \\ -100-y_2 \\ -200+y_2 \end{bmatrix} \\
f(x,y) &:= -y_2 \\
g(x,y) &:= \begin{bmatrix} -x(y_1-2)^2+y_2 \\ -y_2+12.5x(y_1-5) \\ -5(y_1+4-x)(y_1+8-x)+y_2 \end{bmatrix}
\end{aligned}
$$

Comment: Solutions are unknown.

Problem name: `LuDebSinha2016f`

Source: [34]

Description: LuDebSinha2016f is defined as follows

$$
\begin{aligned}
F(x,y) &:= -x_2 \\
G(x,y) &:= \begin{bmatrix} 2-y \\ -4+y \\ -80-x_1 \\ -200+x_1 \\ -100-x_2 \\ -200+x_2 \\ -y\left(\frac{x_1}{20}-2\right)+x_2 \\ -x_2+12.5y\left(\frac{x_1}{20}-5\right) \\ -5\left(\frac{x_1}{20}+4-y\right)\left(\frac{x_1}{20}+8-y\right)+x_2 \end{bmatrix} \\
f(x,y) &:= \left(\frac{x_1-50}{28}\right)^2 + \left(\frac{y-2.5}{0.2}\right)^2
\end{aligned}
$$

Comment: Solutions are unknown.

Problem name: `MacalHurter1997`

Source: [36]

Description: MacalHurter1997 is defined as follows

$$
\begin{aligned}
F(x,y) &:= (x-1)^2+(y-1)^2 \\
f(x,y) &:= 0.5y^2+500y-50xy
\end{aligned}
$$

Comment: The global optimal solution is $(10.0163, 0.8197)$; cf. [36].

Problem name: `Mirrlees1999`

Source: [38]

Description: Mirrlees1999 is defined as follows

$$
\begin{aligned}
F(x,y) &:= (x-2)^2+(y-1)^2 \\
f(x,y) &:= -x\exp\left[-(y+1)^2\right]-\exp\left[-(y-1)^2\right] \\
g(x,y) &:= \begin{bmatrix} -2-y \\ -2+y \end{bmatrix}
\end{aligned}
$$

Comment: We used the version from [55], which added the box constriants on y but the global optimal remains the same. This problem is known in the literature as Mirrlees problem. It is usually used to illustrate how the KKT reformulation of the bilevel optimization problem is not appropriate for problems with nonconvex lower-level problems. The global optimal solution for the problem is $(1, 0.95753)$; [38].

Problem name: `MitsosBarton2006Ex38`

Source: [39]

Description: MitsosBarton2006Ex38 is defined as follows

$$\begin{aligned}
F(x,y) &:= y^2 \\
G(x,y) &:= \begin{bmatrix} -x-1 \\ x-1 \\ -y-0.1 \\ y-0.1 \end{bmatrix} \\
f(x,y) &:= (x+\exp x)y \\
g(x,y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix}
\end{aligned}$$

Comment: The global optimal solution of the problem is $(-0.567, 0)$; cf. [39].

Problem name: `MitsosBarton2006Ex39`

Source: [39]

Description: MitsosBarton2006Ex39 is defined as follows

$$\begin{aligned}
F(x,y) &:= x \\
G(x,y) &:= \begin{bmatrix} -x+y \\ -x+10 \\ x-10 \end{bmatrix} \\
f(x,y) &:= y^3 \\
g(x,y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix}
\end{aligned}$$

Comment: The global optimal solution of the problem is $(-1, -1)$; cf. [39].

Problem name: `MitsosBarton2006Ex310`

Source: [39]

Description: MitsosBarton2006Ex310 is defined as follows

$$\begin{aligned}
F(x,y) &:= y \\
G(x,y) &:= \begin{bmatrix} -x+0.1 \\ x-1 \end{bmatrix} \\
f(x,y) &:= x(16y^4+2y^3-8y^2-1.5y+0.5) \\
g(x,y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix}
\end{aligned}$$

Comment: The set of all global optimal solution is $[0.1,\ 1] \times \{0.5\}$; cf. [39].

Problem name: `MitsosBarton2006Ex311`

Source: [39]

Description: MitsosBarton2006Ex311 is defined as follows

$$\begin{aligned} F(x,y) &:= y \\ G(x,y) &:= \begin{bmatrix} -x-1 \\ x-1 \end{bmatrix} \\ f(x,y) &:= x(16y^4+2y^3-8y^2-1.5y+0.5) \\ g(x,y) &:= \begin{bmatrix} -y-0.8 \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: The global optimal solution of the problem is $(0,-0.8)$; cf. [39].

Problem name: `MitsosBarton2006Ex312`

Source: [39]

Description: MitsosBarton2006Ex312 is defined as follows

$$\begin{aligned} F(x,y) &:= -x+xy+10y^2 \\ G(x,y) &:= \begin{bmatrix} -x-1 \\ x-1 \end{bmatrix} \\ f(x,y) &:= -xy^2+0.5y^4 \\ g(x,y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: The global optimal solution of the problem is $(0,0)$; cf. [39].

Problem name: `MitsosBarton2006Ex313`

Source: [39]

Description: MitsosBarton2006Ex313 is defined as follows

$$\begin{aligned} F(x,y) &:= x-y \\ G(x,y) &:= \begin{bmatrix} -x-1 \\ x-1 \end{bmatrix} \\ f(x,y) &:= 0.5xy^2-x^3y \\ g(x,y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: The global optimal solution of the problem is $(0,1)$; cf. [39].

Problem name: `MitsosBarton2006Ex314`

Source: [39]

Description: MitsosBarton2006Ex314 is defined as follows

$$\begin{aligned} F(x,y) &:= (x-0.25)^2+y^2 \\ G(x,y) &:= \begin{bmatrix} -x-1 \\ x-1 \end{bmatrix} \\ f(x,y) &:= \tfrac{1}{3}y^3-xy \\ g(x,y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: The global optimal solution of the problem is $(0.25,0.5)$; cf. [39].

Problem name: `MitsosBarton2006Ex315`

Source: [39]

Description: MitsosBarton2006Ex315 is defined as follows

$$\begin{aligned} F(x,y) &:= x+y \\ G(x,y) &:= \begin{bmatrix} -x-1 \\ x-1 \end{bmatrix} \\ f(x,y) &:= \tfrac{1}{2}xy^2-\tfrac{1}{3}y^3 \\ g(x,y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: The global optimal solution of the problem is $(-1,1)$; cf. [39].

Problem name: `MitsosBarton2006Ex316`

Source: [39]

Description: MitsosBarton2006Ex316 is defined as follows

$$\begin{aligned} F(x,y) &:= 2x+y \\ G(x,y) &:= \begin{bmatrix} -x-1 \\ x-1 \end{bmatrix} \\ f(x,y) &:= -\tfrac{1}{2}xy^2-\tfrac{1}{4}y^4 \\ g(x,y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: The points $(-1,0)$ and $(-0.5,-1)$ are the two global optimal solutions of the problem; cf. [39].

Problem name: `MitsosBarton2006Ex317`

Source: [39]

Description: MitsosBarton2006Ex317 is defined as follows

$$\begin{aligned} F(x,y) &:= (x+\tfrac{1}{2})^2+\tfrac{1}{2}y^2 \\ G(x,y) &:= \begin{bmatrix} -x-1 \\ x-1 \end{bmatrix} \\ f(x,y) &:= \tfrac{1}{2}xy^2+\tfrac{1}{4}y^4 \\ g(x,y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: The points $(-0.25,0.5)$ and $(-0.25,-0.5)$ are the two global optimal solutions of the problem; cf. [39].

Problem name: `MitsosBarton2006Ex318`

Source: [39]

Description: MitsosBarton2006Ex318 is defined as follows

$$\begin{aligned} F(x,y) &:= -x^2+y^2 \\ G(x,y) &:= \begin{bmatrix} -x-1 \\ x-1 \end{bmatrix} \\ f(x,y) &:= xy^2-\tfrac{1}{2}y^4 \\ g(x,y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: The global optimal solution of the problem is $(0.5,0)$; cf. [39].

Problem name: `MitsosBarton06Ex319`

Source: [39]
Description: MitsosBarton2006Ex319 is defined as follows

$$\begin{aligned} F(x,y) &:= xy - y + \tfrac{1}{2}y^2 \\ G(x,y) &:= \begin{bmatrix} -x-1 \\ x-1 \end{bmatrix} \\ f(x,y) &:= -xy^2 + \tfrac{1}{2}y^4 \\ g(x,y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: The global optimal solution of the problem is $(0.189, 0.4343)$; cf. [39].

Problem name: `MitsosBarton2006Ex320`
Source: [39]
Description: MitsosBarton2006Ex320 is defined as follows

$$\begin{aligned} F(x,y) &:= (x - \tfrac{1}{4})^2 + y^2 \\ G(x,y) &:= \begin{bmatrix} -x-1 \\ x-1 \end{bmatrix} \\ f(x,y) &:= \tfrac{1}{3}y^3 - x^2 y \\ g(x,y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: The global optimal solution of the problem is $(0.5, 0.5)$; cf. [39].

Problem name: `MitsosBarton2006Ex321`
Source: [39]
Description: MitsosBarton2006Ex321 is defined as follows

$$\begin{aligned} F(x,y) &:= (x+0.6)^2 + y^2 \\ G(x,y) &:= \begin{bmatrix} -x-1 \\ x-1 \end{bmatrix} \\ f(x,y) &:= y^4 + \tfrac{4}{30}(-x+1)y^3 + (-0.02x^2 + 0.16x - 0.4)y^2 \\ &\qquad +(0.004x^3 - 0.036x^2 + 0.08x)y \\ g(x,y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: The global optimal solution of the problem is $(-0.5545, 0.4554)$; cf. [39].

Problem name: `MitsosBarton2006Ex322`
Source: [39]
Description: MitsosBarton2006Ex322 is defined as follows

$$\begin{aligned} F(x,y) &:= (x+0.6)^2 + y^2 \\ G(x,y) &:= \begin{bmatrix} -x-1 \\ x-1 \end{bmatrix} \\ f(x,y) &:= y^4 + \tfrac{2}{15}(-x+1)y^3 + (-0.02x^2 + 0.16x - 0.4)y^2 \\ &\qquad +(0.004x^3 - 0.036x^2 + 0.08x)y \\ g(x,y) &:= \begin{bmatrix} -y-1 \\ y-1 \\ 0.01(1+x)^2 - y^2 \end{bmatrix} \end{aligned}$$

Comment: The global optimal solution of the problem is $(-0.5545, 0.4554)$; cf. [39].

Problem name: `MitsosBarton2006Ex323`

Source: [39]

Description: MitsosBarton2006Ex323 is defined as follows

$$\begin{aligned} F(x,y) &:= x^2 \\ G(x,y) &:= \begin{bmatrix} -x-1 \\ x-1 \\ 1+x-9x^2-y \end{bmatrix} \\ f(x,y) &:= y \\ g(x,y) &:= \begin{bmatrix} -y-1 \\ y-1 \\ y^2(x-0.5) \end{bmatrix} \end{aligned}$$

Comment: The global optimal solution of the problem is $(-0.4191, -1)$; cf. [39].

Problem name: `MitsosBarton2006Ex324`

Source: [39]

Description: MitsosBarton2006Ex324 is defined as follows

$$\begin{aligned} F(x,y) &:= x^2-y \\ G(x,y) &:= \begin{bmatrix} -x \\ x-1 \end{bmatrix} \\ f(x,y) &:= \left[(y-1-0.1x)^2-0.5-0.5x\right]^2 \\ g(x,y) &:= \begin{bmatrix} -y \\ y-3 \end{bmatrix} \end{aligned}$$

Comment: The global optimal solution of the problem is $(0.2106, 1.799)$; cf. [39].

Problem name: `MitsosBarton2006Ex325`

Source: [39]

Description: MitsosBarton2006Ex325 is defined as follows

$$\begin{aligned} F(x,y) &:= x_1y_1+x_2y_1^2-x_1x_2y_3 \\ G(x,y) &:= \begin{bmatrix} -x-1_2 \\ x-1_2 \\ 0.1y_1y_2-x_1^2 \\ x_2y_1^2 \end{bmatrix} \\ f(x,y) &:= x_1y_1^2+x_2y_2y_3 \\ g(x,y) &:= \begin{bmatrix} -y-1_3 \\ y-1_3 \\ y_1^2-y_2y_3 \\ y_2^2y_3-y_1x_1 \\ -y_3^2+0.1 \end{bmatrix} \end{aligned}$$

Comment: The best known value for the upper-level objective function is -1 and a corresponding point is $(-1, -1, -1, 1, 1)$; cf. [39].

Problem name: `MitsosBarton2006Ex326`

Source: [39]

Description: MitsosBarton2006Ex326 is defined as follows

$$\begin{aligned} F(x,y) &:= x_1y_1 + x_2y_2^2 + x_1x_2y_3^3 \\ G(x,y) &:= \begin{bmatrix} 0.1 - x_1^2 \\ 1.5 - y_1^2 - y_2^2 - y_3^2 \\ 2.5 + y_1^2 + y_2^2 + y_3^2 \\ -x - 1_2 \\ x - 1_2 \end{bmatrix} \\ f(x,y) &:= x_1y_1^2 + x_2y_2^2 + (x_1 - x_2)y_3^2 \\ g(x,y) &:= \begin{bmatrix} -y - 1_3 \\ y - 1_3 \end{bmatrix} \end{aligned}$$

Comment: The global optimal solution of the problem is $(-1, -1, 1, 1, -0.707)$; cf. [39].

Problem name: `MitsosBarton2006Ex327`

Source: [39]

Description: MitsosBarton2006Ex327 is defined as follows

$$\begin{aligned} F(x,y) &:= \sum_{j=1}^{5}(x_j^2 + y_j^2) \\ G(x,y) &:= \begin{bmatrix} -x - 1_5 \\ x - 1_5 \\ y_1y_2 - x_1 \\ x_2y_1^2 \\ x_1 - \exp x_2 + y_3 \end{bmatrix} \\ f(x,y) &:= y_1^3 + y_2^2x_1 + y_2^2x_2 + 0.1y_3 + (y_4^2 + y_5^2)x_3x_4x_5 \\ g(x,y) &:= \begin{bmatrix} -y - 1_5 \\ y - 1_5 \\ y_1y_2 - 0.3 \\ x_1 - y_3^2 - 0.2 \\ -\exp y_3 + y_4y_5 - 0.1 \end{bmatrix} \end{aligned}$$

Comment: The best known value for the upper-level objective function is 2 and a corresponding point is $(0_5, -1, 0, -1, 0, 0)$; cf. [39].

Problem name: `MitsosBarton2006Ex328`

Source: [39]

Description: MitsosBarton2006Ex328 is defined as follows

$$\begin{aligned} F(x,y) &:= -\sum_{j=1}^{5}(x_j^2 + y_j^2) \\ G(x,y) &:= \begin{bmatrix} -x - 1_5 \\ x - 1_5 \\ y_1y_2 - x_1 \\ x_2y_1^2 \\ x_1 - \exp x_2 + y_3 \end{bmatrix} \\ f(x,y) &:= y_1^3 + y_2^2x_1 + y_2^2x_2 + 0.1y_3 + (y_4^2 + y_5^2)x_3x_4x_5 \\ g(x,y) &:= \begin{bmatrix} -y - 1_5 \\ y - 1_5 \\ y_1y_2 - 0.3 \\ x_1 - y_3^2 - 0.2 \\ -\exp y_3 + y_4y_5 - 0.1 \end{bmatrix} \end{aligned}$$

Comment: The best known values for the upper and lower-level objective functions are -10 and -3.1 respectively, and a corresponding point is $(1, (-1)_5, 1, -1, -1, 1)$; cf. [39]. Another possible solution is $((-1)_5, 1, -1, -1, -1, 1)$ with the same the upper and lower-level objective function values.

Problem name: `MorganPatrone2006a`

Source: [40]

Description: MorganPatrone2006a is defined as follows

$$\begin{array}{rcl} F(x,y) & := & -(x+y) \\ G(x,y) & := & \begin{bmatrix} -x-0.5 \\ x-0.5 \end{bmatrix} \\ f(x,y) & := & xy \\ g(x,y) & := & \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{array}$$

Comment: The global optimal solution is $(0, 1)$; cf. [40].

Problem name: `MorganPatrone2006b`

Source: [40]

Description: MorganPatrone2006b is defined as follows

$$\begin{array}{rcl} F(x,y) & := & -(x+y) \\ f(x,y) & := & \begin{cases} (x+0.25)y & \text{if } x \in [-0.5, -0.25] \\ 0 & \text{if } x \in [-0.25, 0.25] \\ (x-0.25)y & \text{if } x \in [0.25, 0.5] \end{cases} \\ g(x,y) & := & \begin{bmatrix} -0.5-x \\ -0.5+x \\ -1-y \\ -1+y \end{bmatrix} \end{array}$$

Comment: The global optimal solution is $(0.25, 1)$; cf. [40].

Problem name: `MorganPatrone2006c`

Source: [40]

Description: MorganPatrone2006c is defined as follows

$$\begin{array}{rcl} F(x,y) & := & -(x+y) \\ f(x,y) & := & \begin{cases} \left[x + -\frac{7}{4}\right] y & \text{if } x \in \left[-2, -\frac{7}{4}\right] \\ 0 & \text{if } x \in \left[-\frac{7}{4}, \frac{7}{4}\right] \\ \left[x - -\frac{7}{4}\right] y & \text{if } x \in \left[\frac{7}{4}, 2\right] \end{cases} \\ g(x,y) & := & \begin{bmatrix} -x-2 \\ x-2 \\ -y-1 \\ y-1 \end{bmatrix} \end{array}$$

Comment: The global optimal solution is $(2, -1)$; cf. [40].

Problem name: `MuuQuy2003Ex1`

Source: [41]

Description: MuuQuy2003Ex1 is defined as follows

$$\begin{aligned} F(x,y) &:= y_1^2 + y_2^2 + x^2 - 4x \\ G(x,y) &:= \begin{bmatrix} -x \\ x-2 \end{bmatrix} \\ f(x,y) &:= y_1^2 + \tfrac{1}{2}y_2^2 + y_1y_2 + (1-3x)y_1 + (1+x)y_2 \\ g(x,y) &:= \begin{bmatrix} 2y_1 + y_2 - 2x - 1 \\ -y \end{bmatrix} \end{aligned}$$

Comment: The best known solution is $(0.8438, 0.7657, 0)$ according to [41].

Problem name: `MuuQuy2003Ex2`

Source: [41]

Description: MuuQuy2003Ex2 is defined as follows

$$\begin{aligned} F(x,y) &:= y_1^2 + y_3^2 - y_1y_3 - 4y_2 - 7x_1 + 4x_2 \\ G(x,y) &:= \begin{bmatrix} -x \\ x_1 + x_2 - 1 \end{bmatrix} \\ f(x,y) &:= y_1^2 + \tfrac{1}{2}y_2^2 + \tfrac{1}{2}y_3^2 + y_1y_2 + (1-3x_1)y_1 + (1+x_2)y_2 \\ g(x,y) &:= \begin{bmatrix} 2y_1 + y_2 - y_3 + x_1 - 2x_2 + 2 \\ -y \end{bmatrix} \end{aligned}$$

Comment: The best known solution is $(0.609, 0.391, 0.000, 0.000, 1.828)$; cf. [41].

Problem name: `NieEtal2017Ex34`

Source: [42]

Description: NieEtal2017Ex34 is defined as follows

$$\begin{aligned} F(x,y) &:= x + y_1 + y_2 \\ G(x,y) &:= \begin{bmatrix} -x + 2 \\ x - 3 \end{bmatrix} \\ f(x,y) &:= x(y_1 + y_2) \\ g(x,y) &:= \begin{bmatrix} -y_1^2 + y_2^2 + (y_1^2 + y_2^2)^2 \\ -y_1 \end{bmatrix} \end{aligned}$$

Comment: The global optimal solution of the problem is $(2, 0, 0)$; cf. [42].

Problem name: `NieEtal2017Ex52`

Source: [42]

Description: NieEtal2017Ex52 is defined as follows

$$\begin{aligned} F(x,y) &:= x_1y_1 + x_2y_2 + x_1x_2y_1y_2y_3 \\ G(x,y) &:= \begin{bmatrix} -x - 1_2 \\ x - 1_2 \\ y_1y_2 - x_1^2 \end{bmatrix} \\ f(x,y) &:= x_1y_1^2 + x_2^2y_2y_3 - y_1y_3^2 \\ g(x,y) &:= \begin{bmatrix} 1 - y_1^2 - y_2^2 - y_3^2 \\ y_1^2 + y_2^2 + y_3^2 - 2 \end{bmatrix} \end{aligned}$$

Comment: The point $(-1, -1, 1.1097, 0.3143, -0.8184)$ is global optimal solution of the problem provided in [42].

Problem name: `NieEtal2017Ex54`

Source: [42]
Description: NieEtal2017Ex54 is defined as follows

$$
\begin{aligned}
F(x,y) &:= x_1^2y_1 + x_2y_2 + x_3y_3^2 + x_4y_4^2\\
G(x,y) &:= \begin{bmatrix} \|x\|^2-1 \\ y_1y_2 - x_1 \\ y_3y_4 - x_3^2 \end{bmatrix}\\
f(x,y) &:= y_1^2 - y_2(x_1+x_2) - (y_3+y_4)(x_3+x_4)\\
g(x,y) &:= \begin{bmatrix} \|y\|^2-1 \\ y_2^2+y_3^2+y_4^2-y_1 \end{bmatrix}
\end{aligned}
$$

Comment: $(0,-0,-0.7071,-0.7071,0.6180,0,-0.5559,-0.5559)$ is the global optimal solution of the problem obtained in [42].

Problem name: `NieEtal2017Ex57`
Source: [42]
Description: NieEtal2017Ex57 is defined as follows

$$
\begin{aligned}
F(x,y) &:= \tfrac{1}{2}x_1^2y_1 + x_2y_2^2 - (x_1+x_2^2)y_3\\
G(x,y) &:= \begin{bmatrix} -x-1_2 \\ x-1_2 \\ -x_1-x_2+x_1^2+y_1^2+y_2^2 \end{bmatrix}\\
f(x,y) &:= x_2(y_1y_2y_3 + y_2^2 - y_3^3)\\
g(x,y) &:= \begin{bmatrix} -x_1+y_1^2+y_2^2+y_3^2 \\ -1+2y_2y_3 \end{bmatrix}
\end{aligned}
$$

Comment: The point $(1,1,0,0,1)$ is the best known solution of the problem provided in [42].

Problem name: `NieEtal2017Ex58`
Source: [42]
Description: NieEtal2017Ex58 is defined as follows

$$
\begin{aligned}
F(x,y) &:= (x_1+x_2+x_3+x_4)(y_1+y_2+y_3+y_4)\\
G(x,y) &:= \begin{bmatrix} \|x\|^2-1 \\ y_3^2-x_4 \\ y_2y_4-x_1 \end{bmatrix}\\
f(x,y) &:= x_1y_1 + x_2y_2 + 0.1y_3 + 0.5y_4 - y_3y_4\\
g(x,y) &:= \begin{bmatrix} y_1^2+2y_2^2+3y_3^2+4y_4^2-x_1^2-x_3^2-x_2-x_4 \\ -y_2y_3+y_1y_4 \end{bmatrix}
\end{aligned}
$$

Comment: $(0.5135,0.5050,0.4882,0.4929,-0.8346,-0.4104,-0.2106,-0.2887)$ is the best known solution of the problem obtained in [42].

Problem name: `NieEtal2017Ex61`
Source: [42]

Description: NieEtal2017Ex61 is defined as follows

$$\begin{array}{rcl}
F(x,y) & := & y_1^3(x_1^2-3x_1x_2)-y_1^2y_2+y_2x_2^3\\
G(x,y) & := & \begin{bmatrix} -x-1_2\\ x-1_2\\ -y_2-y_1(1-x_1^2)\end{bmatrix}\\
f(x,y) & := & y_1y_2^2-y_2^3-y_1^2(x_2-x_1^2)\\
g(x,y) & := & y_1^2+y_2^2-1
\end{array}$$

Comment: The point $(0.5708, -1, -0.1639, 0.9865)$ is the best known solution of the problem provided in [42].

Problem name: `Outrata1990Ex1a`

Source: [43]

Description: Outrata1990Ex1a is defined as follows

$$\begin{array}{rcl}
F(x,y) & := & \frac{1}{2}(y_1^2+y_2^2)-3y_1-4y_2+r(x_1^2+x_2^2)\\
f(x,y) & := & \frac{1}{2}\langle y, Hy\rangle-\langle b(x), y\rangle\\
g(x,y) & := & \begin{bmatrix} -0.333y_1+y_2-2\\ y_1-0.333y_2-2\\ -y\end{bmatrix}
\end{array}$$

with $r := 0.1$, $H := \begin{bmatrix} 1 & -2\\ -2 & 5\end{bmatrix}$ and $b(x) := x$.

Comment: Outrata1990Ex1b, Outrata1990Ex1c, Outrata1990Ex1d, and Outrata1990Ex1e are obtained by respectively replacing r, H, and $b(x)$ in the lower-level objective function of Outrata1990a by

$$\begin{array}{l}
r := 1, \quad H := \begin{bmatrix} 1 & -2\\ -2 & 5\end{bmatrix}, \quad \text{and} \quad b(x) := x,\\
r := 0, \quad H := \begin{bmatrix} 1 & 3\\ 3 & 10\end{bmatrix}, \quad \text{and} \quad b(x) := x,\\
r := 0.1, \quad H := \begin{bmatrix} 1 & 3\\ 3 & 10\end{bmatrix}, \quad \text{and} \quad b(x) := x,\\
r := 0.1, \quad H := \begin{bmatrix} 1 & 3\\ 3 & 10\end{bmatrix}, \quad \text{and} \quad b(x) := \begin{bmatrix} -1 & 2\\ 3 & -3\end{bmatrix}x.
\end{array}$$

According to [43], the best known solutions for problems Outrata1990Ex1a, Outrata1990Ex1b, Outrata1990Ex1c, Outrata1990Ex1d, and Outrata1990Ex1e are respectively

$$\begin{array}{l}
(0.97, 3.14, 2.6, 1.8), \quad (0.28, 0.48, 2.34, 1.03),\\
(20.26, 42.81, 3, 3), \quad (2, 0.06, 2, 0), \quad \text{and} \quad (2.42, -3.65, 0, 1.58).
\end{array}$$

Note that for Outrata1990Ex1b and Outrata1990Ex1c, the solutions above change with a different starting point for the algorithm used in [43].

Problem name: `Outrata1990Ex2a`

Source: [43]

Description: Outrata1990Ex2a is defined as follows

$$\begin{array}{rcl} F(x,y) & := & \frac{1}{2}\left[(y_1-3)^2+(y_2-4)^2\right] \\ G(x,y) & := & -x \\ f(x,y) & := & \frac{1}{2}\langle y, H(x)y\rangle - (3+1.333x)y_1 - xy_2 \\ g(x,y) & := & \begin{bmatrix} -0.333y_1+y_2-2 \\ y_1-0.333y_2-2 \\ -y \end{bmatrix} \end{array}$$

with the matrix $H(x)$ defined by $H(x) := I$.

Comment: Outrata1990Ex2b, Outrata1990Ex2c, Outrata1990Ex2d and Outrata1990Ex2e are respectively obtained by performing some changes on the terms $H(x)$ and $g(x,y)$ in the lower-level objective function of Outrata1990Ex2a:

$$H(x) := \begin{bmatrix} 1+x & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad g(x,y) := \begin{bmatrix} -0.333y_1+y_2-2 \\ y_1-0.333y_2-2 \\ -y \end{bmatrix},$$

$$H(x) := \begin{bmatrix} 1+x & 0 \\ 0 & 1+0.1x \end{bmatrix}, \quad \text{and} \quad g(x,y) := \begin{bmatrix} -0.333y_1+y_2-2 \\ y_1-0.333y_2-2 \\ -y \end{bmatrix},$$

$$H(x) := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad g(x,y) := \begin{bmatrix} (-0.333+0.1x)y_1+y_2-x \\ y_1+(-0.333-0.1x)y_2-2 \\ -y \end{bmatrix},$$

$$H(x) := \begin{bmatrix} 1+x & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad g(x,y) := \begin{bmatrix} (-0.333+0.1x)y_1+y_2-x \\ y_1+(-0.333-0.1x)y_2-2 \\ -y \end{bmatrix},$$

According to [43], the best known solutions for problems Outrata1990Ex2a, Outrata1990Ex2b, Outrata1990Ex2c, Outrata1990Ex2d, Outrata1990Ex2e are respectively

$$(2.07,3,3), \quad (0,3,3), \quad (3.456,1.707,2.569),$$
$$(2.498,3.632,2.8) \ \text{ and } \ (3.999,1.665,3.887), .$$

Problem name: `Outrata1993Ex31`

Source: [44]

Description: Outrata1993Ex31 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & \frac{1}{2}(y_1-3)^2+\frac{1}{2}(y_2-4)^2 \\ G(x,y) & := & -x \\ f(x,y) & := & \frac{1}{2}(1+0.2x)y_1^2+\frac{1}{2}(1+0.1x)y_2^2-(3+1.33x)y_1-xy_2 \\ g(x,y) & := & \begin{bmatrix} (-0.333+0.1x)y_1+y_2+0.1x-2 \\ y_1+(-0.333-0.1x)y_2+0.1x-2 \\ -y \end{bmatrix} \end{array}$$

Comment: `Outrata1993Ex32` is obtained by replacing the lower-level constraint by

$$g(x,y) \ := \ \begin{bmatrix} -0.333y_1+y_2+0.1x-1 \\ y_1^2+y_2^2-0.1x-9 \\ -y \end{bmatrix}.$$

For `Outrata1993Ex1` and `Outrata1993Ex2`, the best known solutions from [44] are $(1.90910, 2.97836, 2.23182)$ and $(4.06095, 2.68227, 1.48710)$, respectively.

Problem name: `Outrata1994Ex31`

Source: [45]
Description: Outrata1994Ex31 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & \frac{1}{2}(y_1-3)^2+\frac{1}{2}(y_2-4)^2 \\ G(x,y) & := & \begin{bmatrix} -x \\ x-10 \end{bmatrix} \\ f(x,y) & := & \frac{1}{2}(1+0.2x)y_1^2+\frac{1}{2}(1+0.1x)y_2^2-(3+1.333x)y_1-xy_2 \\ g(x,y) & := & \begin{bmatrix} -0.333y_1+y_2+0.1x-1 \\ y_1^2+y_2^2-0.1x-9 \\ -y \end{bmatrix} \end{array}$$

Comment: According to [45], the best known solution is $(4.0604, 2.6822, 1.4871)$.

Problem name: `OutrataCervinka2009`
Source: [46]
Description: OutrataCervinka2009 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & -2x_1-0.5x_2-y_2 \\ G(x,y) & := & x_1 \\ f(x,y) & := & y_1-y_2+x^\top y+\frac{1}{2}y^\top y \\ g(x,y) & := & \begin{bmatrix} y_2 \\ y_2-y_1 \\ y_2+y_1 \end{bmatrix} \end{array}$$

Comment: The point 0_4 is the global optimal solution of the problem according to [46].

Problem name: `PaulaviciusEtal2017a`
Source: [47]
Description: PaulaviciusEtal2017a is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x^2+y^2 \\ G(x,y) & := & \begin{bmatrix} -x-1 \\ x-1 \\ -y-1 \\ y-1 \end{bmatrix} \\ f(x,y) & := & xy^2-\frac{1}{2}y^4 \\ g(x,y) & := & \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{array}$$

Comment: This problem a slight modification of MitsosBarton2006Ex318, just with the upper-level objective function there replaced by x^2+y^2. Doing so, the point $(0.5, 0)$ remains globally optimal for the new problem [47].

Problem name: `PaulaviciusEtal2017b`
Source: [47]

Description: PauláviciusEtal2017b is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x+y \\ G(x,y) & := & \begin{bmatrix} -x-1 \\ x-1 \\ -y-1 \\ y-1 \end{bmatrix} \\ f(x,y) & := & 0.5xy^2 - x^3y \\ g(x,y) & := & \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{array}$$

Comment: This problem a slight modification of MitsosBarton2006Ex313, just with the *minus* in upper-level objective function replaced by a *plus*. Doing so, the global optimal solution the new problem above is $(-1,-1)$ according to [47].

Problem name: `SahinCiric1998Ex2`

Source: [48]

Description: SahinCiric1998Ex2 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & (x-3)^2 + (y-2)^2 \\ G(x,y) & := & \begin{bmatrix} -x \\ x-8 \end{bmatrix} \\ f(x,y) & := & (y-5)^2 \\ g(x,y) & := & \begin{bmatrix} -2x+y-1 \\ x-2y+2 \\ x+2y-14 \end{bmatrix} \end{array}$$

Comment: The optimal value for the upper-level objective function is 5 and a corresponding global optimal point is $(1,3)$; cf. [48].

Problem name: `ShimizuAiyoshi1981Ex1`

Source: [49]

Description: ShimizuAiyoshi1981Ex1 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x^2 + (y-10)^2 \\ G(x,y) & := & \begin{bmatrix} -15+x \\ y-x \\ -x \end{bmatrix} \\ f(x,y) & := & (x+2y-30)^2 \\ g(x,y) & := & \begin{bmatrix} x+y-20 \\ y-20 \\ -y \end{bmatrix} \end{array}$$

Comment: The global optimal solution of the problem is $(10,10)$ according to [49].

Problem name: `ShimizuAiyoshi1981Ex2`

Source: [49]

Description: ShimizuAiyoshi1981Ex2 is defined as follows

$$\begin{aligned} F(x,y) &:= (x_1-30)^2+(x_2-20)^2-20y_1+20y_2 \\ G(x,y) &:= \begin{bmatrix} -x_1-2x_2+30 \\ x_1+x_2-25 \\ x_2-15 \end{bmatrix} \\ f(x,y) &:= (x_1-y_1)^2+(x_2-y_2)^2 \\ g(x,y) &:= \begin{bmatrix} y-10_2 \\ -y \end{bmatrix} \end{aligned}$$

Comment: The global optimal solution of the problem is $(20,5,10,5)$ according to [49].

Problem name: `ShimizuEtal1997a`

Source: [50]

Description: ShimizuEtal1997a is defined as follows

$$\begin{aligned} F(x,y) &:= (x-5)^2+(2y+1)^2 \\ f(x,y) &:= (y-1)^2-1.5xy \\ g(x,y) &:= \begin{bmatrix} -3x+y+3 \\ x-0.5y-4 \\ x+y-7 \end{bmatrix} \end{aligned}$$

Comment: Solutions are unknown. A possible solution is $(5,2)$.

Problem name: `ShimizuEtal1997b`

Source: [50]

Description: ShimizuEtal1997b is defined as follows

$$\begin{aligned} F(x,y) &:= 16x^2+9y^2 \\ G(x,y) &:= \begin{bmatrix} -4x+y \\ -x \end{bmatrix} \\ f(x,y) &:= (x+y-20)^4 \\ g(x,y) &:= \begin{bmatrix} 4x+y-50 \\ -y \end{bmatrix} \end{aligned}$$

Comment: $(11.25,5)$ is the global optimal solution of the problem and $(7.2,12.8)$ is a local optimal solution [50].

Problem name: `SinhaMaloDeb2014TP3`

Source: [51]

Description: SinhaMaloDeb2014TP3 is defined as follows

$$\begin{aligned} F(x,y) &:= -x_1^2-3x_2^2-4y_1+y_2^2 \\ G(x,y) &:= \begin{bmatrix} x_1^2+2x_2-4 \\ -x \end{bmatrix} \\ f(x,y) &:= 2x_1^2+y_1^2-5y_2 \\ g(x,y) &:= \begin{bmatrix} -x_1^2+2x_1-x_2^2+2y_1-y_2-3 \\ -x_2-3y_1+4y_2+4 \\ -y \end{bmatrix} \end{aligned}$$

Comment: The best known values of the upper-level and lower-level objective values are -18.6787 and -1.0156, respectively; cf. [51].

Problem name: `SinhaMaloDeb2014TP6`

Source: [51]

Description: SinhaMaloDeb2014TP6 is defined as follows

$$\begin{aligned} F(x,y) &:= (x-1)^2 + 2y_1 - 2x \\ G(x,y) &:= -x \\ f(x,y) &:= (2y_1-4)^2 + (2y_2-1)^2 + xy_1 \\ g(x,y) &:= \begin{bmatrix} 4x + 5y_1 + 4y_2 - 12 \\ 4y_2 - 4x - 5y_1 + 4 \\ 4x - 4y_1 + 5y_2 - 4 \\ 4y_1 - 4x + 5y_2 - 4 \\ -y \end{bmatrix} \end{aligned}$$

Comment: The best known values of the upper-level and lower-level objective values are -1.2091 and 7.6145, respectively; cf. [51].

Problem name: `SinhaMaloDeb2014TP7`

Source: [51]

Description: SinhaMaloDeb2014TP7 is defined as follows

$$\begin{aligned} F(x,y) &:= -\frac{(x_1+y_1)(x_2+y_2)}{1+x_1y_1+x_2y_2} \\ G(x,y) &:= \begin{bmatrix} x_1^2 + x_2^2 - 100 \\ x_1 - x_2 \\ -x \end{bmatrix} \\ f(x,y) &:= \frac{(x_1+y_1)(x_2+y_2)}{1+x_1y_1+x_2y_2} \\ g(x,y) &:= \begin{bmatrix} y - x \\ -y \end{bmatrix} \end{aligned}$$

Comment: The best known values of the upper-level and lower-level objective values are -1.96 and 1.96, respectively; cf. [51].

Problem name: `SinhaMaloDeb2014TP8`

Source: [51]

Description: SinhaMaloDeb2014TP8 is defined as follows

$$\begin{aligned} F(x,y) &:= |2x_1 + 2x_2 - 3y_1 - 3y_2 - 60| \\ g(x,y) &:= \begin{bmatrix} x_1 + x_2 + y_1 - 2y_2 - 40 \\ x - 50_2 \\ -x \end{bmatrix} \\ f(x,y) &:= (y_1 - x_1 + 20)^2 + (y_2 - x_2 + 20)^2 \\ g(x,y) &:= \begin{bmatrix} 2y - x + 10_2 \\ y - 20_2 \\ -y - 10_2 \end{bmatrix} \end{aligned}$$

Comment: The global optimal values of the upper-level and lower-level objective values are 0 and 100.0, respectively; cf. [51].

Problem name: `SinhaMaloDeb2014TP9`

Source: [51]

Description: SinhaMaloDeb2014TP9 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & \sum_{i=1}^{10}\left[(x_i-1)^2+y_i^2\right] \\ f(x,y) & := & \exp\left\{\left[1+\frac{1}{400}\sum_{i=1}^{10}y_i^2-\prod_{i=1}^{10}\cos\left(\frac{y_i}{\sqrt{i}}\right)\right]\sum_{i=1}^{10}x_i^2\right\} \\ g(x,y) & := & \begin{bmatrix} y-\pi_{10} \\ -y-\pi_{10} \end{bmatrix} \end{array}$$

Comment: The best known values of the upper-level and lower-level objective values are 0.0 and 1.0, respectively; cf. [51].

Problem name: `SinhaMaloDeb2014TP10`

Source: [51]

Description: SinhaMaloDeb2014TP10 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & \sum_{i=1}^{10}\left[(x_i-1)^2+y_i^2\right] \\ f(x,y) & := & \exp\left[1+\frac{1}{4000}\sum_{i=1}^{10}x_i^2y_i^2-\prod_{i=1}^{10}\cos\left(\frac{x_iy_i}{\sqrt{i}}\right)\right] \\ g(x,y) & := & \begin{bmatrix} y-\pi_{10} \\ -y-\pi_{10} \end{bmatrix} \end{array}$$

Comment: The best known values of the upper-level and lower-level objective values are 0.0 and 1.0, respectively; cf. [51].

Problem name: `TuyEtal2007`

Source: [52]

Description: TuyEtal2007 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x^2+y^2 \\ G(x,y) & := & \begin{bmatrix} -x \\ -y \end{bmatrix} \\ f(x,y) & := & -y \\ g(x,y) & := & \begin{bmatrix} 3x+y-15 \\ x+y-7 \\ x+3y-15 \end{bmatrix} \end{array}$$

Comment: The global optimal solution of the problem is $(4.492188, 1.523438)$; cf. [52].

Problem name: `Vogel2002`

Source: [53]

Description: Vogel2002 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & (y+1)^2 \\ G(x,y) & := & \begin{bmatrix} -x-3 \\ x-2 \end{bmatrix} \\ f(x,y) & := & y^3-3y \\ g(x,y) & := & x-y \end{array}$$

Comment: The point $(-2,-2)$ is the global optimal solution of the problem; cf. [53].

Problem name: `WanWangLv2011`

Source: [54]

Description: WanWangLv2011 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & (1+x_1-x_2+2y_2)(8-x_1-2y_1+y_2+5y_3) \\ f(x,y) & := & 2y_1-y_2+y_3 \\ g(x,y) & := & \begin{bmatrix} -y_1+y_2+y_3-1 \\ 2x_1-y_1+2y_2-0.5y_3-1 \\ 2x_2+2y_1-y_2-0.5y_3-1 \\ -x \\ -y \end{bmatrix} \end{array}$$

Comment: The global optimal solution is $(0, 0.75, 0, 0.5, 0)$ according to [54].

Problem name: `YeZhu2010Ex42`

Source: [55]

Description: YeZhu2010Ex42 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & (x-1)^2+y^2 \\ G(x,y) & := & \begin{bmatrix} -x-3 \\ x-2 \end{bmatrix} \\ f(x,y) & := & y^3-3y \\ g(x,y) & := & x-y \end{array}$$

Comment: YeZhu2010Ex43 is obtained by replacing the upper-level objective function by

$$(x-1)^2+(y-2)^2$$

Note that YeZhu2010Ex42 is a slightly modified version of Vogel2002, with the term y^2 added to the upper-level objective function. The point $(1, 1)$ is the global optimal solution for both YeZhu2010Ex42 and YeZhu2010Ex43; cf. [55].

Problem name: `Yezza1996Ex31`

Source: [56]

Description: Yezza1996Ex31 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & -(4x-3)y+2x+1 \\ G(x,y) & := & \begin{bmatrix} -x \\ x-1 \end{bmatrix} \\ f(x,y) & := & -(1-4x)y-2x-2 \\ g(x,y) & := & \begin{bmatrix} -y \\ y-1 \end{bmatrix} \end{array}$$

Comment: The global optimal solution of the problem is $(0.25, 0)$; cf. [56].

Problem name: `Yezza1996Ex41`

Source: [56]

Description: Yezza1996Ex41 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & \frac{1}{2}(y-2)^2+\frac{1}{2}(x-y-2)^2 \\ f(x,y) & := & \frac{1}{2}y^2+x-y \\ g(x,y) & := & \begin{bmatrix} -y \\ y-x \end{bmatrix} \end{array}$$

Comment: The global optimal solution of the problem is $(3, 1)$; cf. [56].

Problem name: `Zlobec2001a`

Source: [57]

Description: Zlobec2001a is defined as follows

$$\begin{aligned} F(x,y) &:= -y_1/x \\ f(x,y) &:= -y_1 - y_2 \\ g(x,y) &:= \begin{bmatrix} x + y_1 \\ y_2 - 1 \\ -y \end{bmatrix} \end{aligned}$$

Comment: This example is used in [57] to illustrate that the objective function of the problem can be discontinuous. As stated in [57] , a global optimal solution is $(1, 1, 0)$.

Problem name: `Zlobec2001b`

Source: [57]

Description: Zlobec2001b is defined as follows

$$\begin{aligned} F(x,y) &:= x + y \\ G(x,y) &:= \begin{bmatrix} x - 1 \\ -x \end{bmatrix} \\ f(x,y) &:= -y \\ g(x,y) &:= \begin{bmatrix} y - 1 \\ -y \\ xy \\ -xy \end{bmatrix} \end{aligned}$$

Comment: This example is used in [57] to illustrate that the feasible set of a bilevel optimization problem is not necessarily closed. As stated in [57], this problem does not have an optimal solution.

Problem name: `DesignCentringP1`

Source: [71]

Description: DesignCentringP1 is a so-called design centring problem. The following model is built from Problem 1 in [71, Section 5.3] but with different G, f, g and y.

$$\begin{aligned} F(x,y) &:= -\pi x_3^2 \\ G(x,y) &:= \begin{bmatrix} -y_1 - y_2^2 \\ \frac{1}{4}y_3 + y_4 - \frac{3}{4} \\ -y_6 - 1 \end{bmatrix} \\ f(x,y) &:= y_1 + y_2^2 - \frac{1}{4}y_3 - y_4 + y_6 \\ g(x,y) &:= \begin{bmatrix} (y_1 - x_1)^2 + (y_2 - x_2)^2 - x_3^2 \\ (y_3 - x_1)^2 + (y_4 - x_2)^2 - x_3^2 \\ (y_5 - x_1)^2 + (y_6 - x_2)^2 - x_3^2 \end{bmatrix} \end{aligned}$$

Comment: We construct the lower level objectve function by $f(x,y) = -\sum_i G_i(x,y)$. The reported optimal value of the upper-level objective is $F(x,y) = 1.8606$; cf. [71]. However, since the whole problem is altered, the optimal value of the upper-level objective might be different. A possible solution is

$$(0.7486, -0.2304, 0.7696, -0.0084, -0.0917, 0.9352, 0.5162, 0.7486, -1)$$

with $F(x,y) = -1.4319$, $f(x,y) = -1.7500$.

Problem name: `DesignCentringP2`

Source: [71]

Description: DesignCentringP2 is built from Problem 2 in [71, Section 5.3] but with different G, f, g and y.

$$
\begin{aligned}
F(x,y) &:= -\pi x_3 x_4 \\
G(x,y) &:= \begin{bmatrix} -y_1 - y_2^2 \\ \frac{1}{4}y_3 + y_4 - \frac{3}{4} \\ -y_6 - 1 \\ x_3 - 1 \\ x_4 - 1 \end{bmatrix} \\
f(x,y) &:= y_1 + y_2^2 - \frac{1}{4}y_3 - y_4 + y_6 \\
g(x,y) &:= \begin{bmatrix} \frac{(y_1-x_1)^2}{x_3^2} + \frac{(y_2-x_2)^2}{x_4^2} - 1 \\ \frac{(y_3-x_1)^2}{x_3^2} + \frac{(y_4-x_2)^2}{x_4^2} - 1 \\ \frac{(y_5-x_1)^2}{x_3^2} + \frac{(y_6-x_2)^2}{x_4^2} - 1 \end{bmatrix}
\end{aligned}
$$

Comment: We construct the lower level objectve function by $f(x,y) = -\sum_i G_i(x,y)$. The reported optimal value of the upper-level objective is $F(x,y) = 3.4838$; cf. [71]. However, since the whole problem is altered, the optimal value of the upper-level objective might be different. A possible solution is $(3,0,1,1,3,0,3,0,3,0)$ with $F(x,y) = -\pi$, $f(x,y) = 2.25$.

Problem name: `DesignCentringP3`

Source: [71]

Description: DesignCentringP3 is built from Problem 3 in [71, Section 5.3] but with different G, f, g and y.

$$
\begin{aligned}
F(x,y) &:= -\pi \left| \det \begin{bmatrix} x_3 & x_4 \\ x_5 & x_6 \end{bmatrix} \right| \\
G(x,y) &:= \begin{bmatrix} -y_1 - y_2^2 \\ \frac{1}{4}y_3 + y_4 - \frac{3}{4} \\ -y_6 - 1 \end{bmatrix} \\
f(x,y) &:= y_1 + y_2^2 - \frac{1}{4}y_3 - y_4 + y_6 \\
g(x,y) &:= \begin{bmatrix} \begin{bmatrix} y_1 - x_1 \\ y_2 - x_2 \end{bmatrix}^\top \left(\begin{bmatrix} x_3 & x_4 \\ x_5 & x_6 \end{bmatrix} \begin{bmatrix} x_3 & x_5 \\ x_4 & x_6 \end{bmatrix} \right)^{-1} \begin{bmatrix} y_1 - x_1 \\ y_2 - x_2 \end{bmatrix} - 1 \\ \begin{bmatrix} y_3 - x_1 \\ y_4 - x_2 \end{bmatrix}^\top \left(\begin{bmatrix} x_3 & x_4 \\ x_5 & x_6 \end{bmatrix} \begin{bmatrix} x_3 & x_5 \\ x_4 & x_6 \end{bmatrix} \right)^{-1} \begin{bmatrix} y_3 - x_1 \\ y_4 - x_2 \end{bmatrix} - 1 \\ \begin{bmatrix} y_5 - x_1 \\ y_6 - x_2 \end{bmatrix}^\top \left(\begin{bmatrix} x_3 & x_4 \\ x_5 & x_6 \end{bmatrix} \begin{bmatrix} x_3 & x_5 \\ x_4 & x_6 \end{bmatrix} \right)^{-1} \begin{bmatrix} y_5 - x_1 \\ y_6 - x_2 \end{bmatrix} - 1 \end{bmatrix}
\end{aligned}
$$

Comment: We construct the lower level objectve function by $f(x,y) = -\sum_i G_i(x,y)$. The reported optimal value of the upper-level objective is $F(x,y) = 3.7234$; cf. [71]. However, since the whole problem is altered, the optimal value of the upper-level objective might be different. This problem is a very challenging one whose first, second order derivatives of g are very complicated.

Problem name: `DesignCentringP4`

Source: [71]

Description: DesignCentringP4 is built from Problem 4 in [71, Section 5.3] but with different G, f, g and y.

$$\begin{aligned}
F(x,y) &:= -(x_1-x_3)(x_2-x_4)\\
G(x,y) &:= \begin{bmatrix} -y_1-y_2^2 \\ \frac{1}{4}y_3+y_4-\frac{3}{4} \\ -y_6-1 \end{bmatrix}\\
f(x,y) &:= y_1+y_2^2-\frac{1}{4}y_3-y_4+y_6\\
g(x,y) &:= \begin{bmatrix} y-[x_1,x_2,x_1,x_2,x_1,x_2]^\top \\ -y+[x_3,x_4,x_3,x_4,x_3,x_4]^\top \end{bmatrix}
\end{aligned}$$

Comment: We construct the lower level objectve function by $f(x,y)=-\sum_i G_i(x,y)$. The reported optimal value of the upper-level objective is $F(x,y)=3.0792$; cf. [71]. However, since the whole problem is altered, the optimal value of the upper-level objective might be different. A possible solution is $(-1,1,-1,1,-1,1,-1,1,-1,1)$ with $F(x,y)=0, f(x,y)=0.25$.

Problem name: `OptimalControl`

Source: [37]

Description: `OptimalControl` is a bilevel optimal control program from [37]. This is a quadratic program and its dimensions $\{n_x,n_y,n_G,n_g\}$ are able to be altered. The model below is constructed based on the MATLAB M-file that the author P. Mehlitz in [37] shared with us.

$$\begin{aligned}
F(x,y) &:= \frac{1}{2}\left(\begin{bmatrix} y^1 \\ 0 \end{bmatrix}-c\right)^\top M\left(\begin{bmatrix} y^1 \\ 0 \end{bmatrix}-c\right)-k^\top x,\\
G(x,y) &:= \begin{bmatrix} -x_1+x_2-1 \\ -x \end{bmatrix},\\
f(x,y) &:= \frac{1}{2}(Cy^1-Px)^\top W(Cy^1-Px)+\frac{\sigma}{2}(y^2-Qx)^\top U(y^2-Qx),\\
g(x,y) &:= \begin{bmatrix} y^2-u \\ -y^2+l \\ Ay \\ -Ay \end{bmatrix}.
\end{aligned}$$

Comment: $y^\top=((y^1)^\top\ (y^2)^\top)^\top$ with $y^i\in\mathbb{R}^{m_i}$ and $m_1+m_2=n_y$, $M\in\mathbb{R}^{m\times m}, c\in\mathbb{R}^m, k\in\mathbb{R}^{n_x}, C\in\mathbb{R}^{s\times m_1}, P\in\mathbb{R}^{s\times n}, W\in\mathbb{R}^{s\times s}, Q\in\mathbb{R}^{m_2\times n_x}, U\in\mathbb{R}^{m_2\times m_2}, u\in\mathbb{R}^{m_2}, l\in\mathbb{R}^{m_2}, A\in\mathbb{R}^{t\times n_y}$ are given data. Particularly, in the MATLAB M-file, $n_x=2, n_y=t=2\texttt{ni}, n_G=3, n_g=4\texttt{ni}$, $m_1=m_2=s=\texttt{ni}$, where `ni` and $m>\texttt{ni}$ are two large numbers.

Problem name: `NetworkDesignP1`

Source: [11]

Description: NetworkDesignP1 is a network design problem from NDP1 in [11, Section 4.5] defined as follows

$$\begin{array}{rcl} F(x,y) & := & \left[50+\frac{y_1}{1+x_1}, 10\frac{y_2}{1+x_2}, 10+\frac{y_3}{1+x_3}, 10\frac{y_4}{1+x_4}, 50+\frac{x_5}{1+x_5}\right] y + 100\sum_{i=1}^{5} x_i \\ G(x,y) & := & -x-1 \\ f(x,y) & := & \int_0^{y_1}\left(50+\frac{u}{1+x_1}\right)du + \int_0^{y_2}\left(10\frac{u}{1+x_2}\right)du + \int_0^{y_3}\left(10+\frac{u}{1+x_3}\right)du \\ & & +\int_0^{y_4}\left(10\frac{u}{1+x_4}\right)du + \int_0^{y_5}\left(50+\frac{u}{1+x_5}\right)du \\ g(x,y) & := & \begin{bmatrix} \phi(x,y) \\ -\phi(x,y) \\ -y \end{bmatrix} \text{ where } \phi(x,y) := \begin{bmatrix} -6+y_1+y_3+y_5 \\ y_2-y_5-y_3 \\ y_4-y_1-y_3 \end{bmatrix} \end{array}$$

Comment: The best known values of upper-level and lower-level objectives are $F(x,y) = 300.5$ and $f(x,y) = 419.8$; cf. [8].

Problem name: NetworkDesignP2

Source: [11]

Description: NetworkDesignP2 is a network design problem from NDP2 in [11, Section 4.5] defined as follows

$$\begin{array}{rcl} F(x,y) & := & \left[50+\frac{y_1}{1+x_1}, 10\frac{y_2}{1+x_2}, 10+\frac{y_3}{1+x_3}, 10\frac{y_4}{1+x_4}, 50+\frac{x_5}{1+x_5}\right] y + \sum_{i=1}^{5} x_i \\ G(x,y) & := & -x-1 \\ f(x,y) & := & \int_0^{y_1}\left(50+\frac{u}{1+x_1}\right)du + \int_0^{y_2}\left(10\frac{u}{1+x_2}\right)du + \int_0^{y_3}\left(10+\frac{u}{1+x_3}\right)du \\ & & +\int_0^{y_4}\left(10\frac{u}{1+x_4}\right)du + \int_0^{y_5}\left(50+\frac{u}{1+x_5}\right)du \\ g(x,y) & := & \begin{bmatrix} \phi(x,y) \\ -\phi(x,y) \\ -y \end{bmatrix} \text{ with } \phi(x,y) := \begin{bmatrix} -6+y_1+y_3+y_5 \\ y_2-y_5-y_3 \\ y_4-y_1-y_3 \end{bmatrix} \end{array}$$

Comment: For Network2 the best known values of upper-level and lower-level objectives are $F(x,y) = 142.9$ and $f(x,y) = 81.95$; cf. [8].

Problem name: RobustPortfolioP1

Source: [71]

Description: RobustPortfolioP1 is a portfolio problem which has been used in [71] to illustrate the robustness of an optimization problem. Based on Problem 6 in [71, Section 5.3], we build (1.2) in the following form

$$\begin{array}{rcll} F(x,y) & := & -x_{N+1} & \\ G(x,y) & := & \begin{bmatrix} x_{N+1} - y^\top x \\ -x_{1:N} \\ \sum_{i=1}^{N} x_i - 1 \\ -\sum_{i=1}^{N} x_i + 1 \end{bmatrix} & \\ f_j(x,y^j) & := & y^\top x_{1:N} - x_{N+1} & \\ g(x,y) & := & \begin{bmatrix} \|\text{diag}\,(\sigma)^{-1}(y-\bar{y})\|_\delta^\delta - \theta^\delta \\ -y \end{bmatrix} & \\ \bar{y}_i & := & 1.15 + \frac{0.05}{N} i & \text{for} \quad i = 1, ..., N \\ \sigma_i & := & \frac{0.05}{3N}\sqrt{2N(N+1)i} & \text{for} \quad i = 1, ..., N \\ \theta & := & 1.5 & \\ \delta & \in & [1, \infty] & \end{array}$$

Comment: Here, $\|y\|_\delta^\delta = (\sum_i |y_i|^\delta)^{1/\delta}$ for $\delta \in [1, +\infty]$. The upper-level variable is $x = (x_{1:N}; x_{N+1})$ with $x_{1:N} \in \mathbb{R}^N$. The scenarios considered for N are $N = 10, 50, 100, 150$. The equality constraint $H(x, y) := \sum_{i=1}^N x_i - 1$ is moved to $G(x, y)$.

Comment2: When $\delta = 2$, the optimal solution is $x_i = 1/N, y_i = 1.15, i = 1, \cdots, N, x_{N+1} = 1.15$. This is same as Problem 5 in [71].

Problem name: `RobustPortfolioP2`

Source: [71]

Description: RobustPortfolioP2 is built from Problem 7 in [71, Section 5.3]. It has the form as

$$
\begin{array}{rcl}
F(x,y) & := & -x_{N+1} \\
G(x,y) & := & \begin{bmatrix} x_{N+1} - y^\top x \\ -x_{1:N} \\ \sum_{i=1}^N x_i - 1 \\ -\sum_{i=1}^N x_i + 1 \end{bmatrix} \\
f(x,y) & := & y^\top x_{1:N} - x_{N+1} \\
g(x,y) & := & \begin{bmatrix} \sum_{i=1}^N \frac{(y_i - \bar{y}_i)^2}{\sigma_i^2} - \left\{\theta\left[1 + \sum_{i=1}^N \left(x_i - \frac{1}{N}\right)^2\right]\right\}^2 \\ -y \end{bmatrix} \\
\bar{y}_i & := & 1.15 + \frac{0.05}{N} i \qquad \text{for} \quad i = 1, ..., N \\
\sigma_i & := & \frac{0.05}{3N}\sqrt{2N(N+1)i} \quad \text{for} \quad i = 1, ..., N \\
\theta & := & 1.5
\end{array}
$$

Comment: $N = 10, 50, 100, 150$. A possible solution is $x_i = 1/N, y_i = 1.15, i = 1, \cdots, N$ and $x_{N+1} = 1.15$.

Problem name: `TollSettingP1`

Source: [11]

Description: TollSettingP1 is a toll-setting problem which is able to be defined as follows

$$
\begin{array}{rcl}
F(x,y) & := & -(x_1 y_3 + x_2 y_4 + x_3 y_8) \\
G(x,y) & := & -x \\
f(x,y) & := & [2, 6, 5 + x_1, x_2, 4, 2, 6, x_3] y \\
g(x,y) & := & \begin{bmatrix} \phi(x,y) \\ -\phi(x,y) \\ -y \end{bmatrix} \text{ with } \phi(x,y) := \begin{bmatrix} y_1 + y_2 + y_3 - 1 \\ y_4 + y_5 - y_1 \\ y_6 + y_7 - y_2 - y_4 \\ y_8 - y_5 - y_6 \\ y_3 + y_7 + y_8 - 1 \end{bmatrix}
\end{array}
$$

Comment: The best known values of F and F are $F(x, y) = -7$ and $f(x, y) = 12$, cf. [8], and $(7, 4, 6, 0, 0, 1, 0, 0, 0, 0, 0)$ is a possible solution.

Problem name: `TollSettingP2`

Source: [11]

Description: TollSettingP2 is defined as follows

$$
\begin{aligned}
F(x,y) &:= -(x_1(y_1+y_2)+x_2(y_3+y_4)+x_3(y_5+y_6))\\
G(x,y) &:= -x\\
f(x,y) &:= [2x_1,2x_1,2x_2,2x_2,2x_3,2x_3,5,7,14,7,2,4,29,20,12,8,5,2]y\\
g(x,y) &:= \begin{bmatrix} \phi(x,y)\\ -\phi(x,y)\\ -y \end{bmatrix} \text{ with } \phi(x,y) := \begin{bmatrix} y_7+y_8+y_9-1\\ y_{10}+y_{11}+y_{12}-1\\ y_{13}+y_{14}+y_{15}-1\\ y_{16}+y_{17}+y_{18}-1\\ y_1+y_5+y_{13}-y_7\\ y_2+y_6+y_{16}-y_{10}\\ y_3+y_{14}-y_1-y_8\\ y_4+y_{17}-y_2-y_{11}\\ y_{15}-y_3-y_5-y_9\\ y_{18}-y_4-y_6-y_{12} \end{bmatrix}
\end{aligned}
$$

Comment: The best known values of upper-level and lower-level objectives are $F(x,y) = -9$ and $f(x,y) = 32$; cf. [8]. $(0.5,4,4.5,0,0,0,0,1,0,1,0,0,0,0,1,0,0,1,0,0,1)$ is a possible solution but with $F = -4.5$ and $f = 32$.

Problem name: `TollSettingP3`

Source: [11]

Description: TollSettingP3 is defined as follows

$$
\begin{aligned}
F(x,y) &:= -(x_1(y_1+10y_2)+x_2(y_3+10y_4)+x_3(y_5+10y_6))\\
G(x,y) &:= -x\\
f(x,y) &:= [2x_1,20x_1,2x_2,20x_2,2x_3,20x_3,5,7,14,7,2,4,29,20,12,8,5,2]y\\
g(x,y) &:= \begin{bmatrix} \phi(x,y)\\ -\phi(x,y)\\ -y \end{bmatrix} \text{ with } \phi(x,y) := \begin{bmatrix} y_7+y_8+y_9-1\\ y_{10}+y_{11}+y_{12}-1\\ y_{13}+y_{14}+y_{15}-1\\ y_{16}+y_{17}+y_{18}-1\\ y_1+y_5+y_{13}-y_7\\ 10y_2+10y_6+y_{16}-y_{10}\\ y_3+y_{14}-y_1-y_8\\ 10y_4+y_{17}-10y_2-y_{11}\\ y_{15}-y_3-y_5-y_9\\ y_{18}-10y_4-10y_6-y_{12} \end{bmatrix}
\end{aligned}
$$

Comment: The best known values of upper-level and lower-level objectives are $F(x,y) = -24$ and $f(x,y) = 81$; cf. [8]. $(5,3.5,8.5,0,0,1,0,0,0,0,1,0,0,0,1,0,0,1,0,0,1)$ is a possible solution but with $F = -3.5$ and $f = 32$.

Problem name: `TollSettingP4`

Source: [11]

Description: TollSettingP4 is defined as follows

$$
\begin{aligned}
F(x,y) &:= -(y_2+y_3)x_1-y_3x_2\\
f(x,y) &:= [8,3+2x_1,3+2x_1+2x_2,6]y\\
g(x,y) &:= \begin{bmatrix} \phi(x,y)\\ -\phi(x,y)\\ -y \end{bmatrix} \text{ with } \phi(x,y) := \begin{bmatrix} y_1+y_2-1\\ y_3+y_4-1 \end{bmatrix}
\end{aligned}
$$

Comment: The best known values of upper-level and lower-level objectives are $F(x, y) = -8$ and $f(x, y) = 14$; cf. [8]. Two possible optimal solutions are $(2.5, -1, 0, 1, 1, 0)$ and $(10/3, -4/3, 1, 0, 0, 1)$ but both with $F = -4$, $f = 14$.

Problem name: `TollSettingP5`

Source: [11]

Description: TollSettingP5 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & -(y_2+y_3)x \\ f(x,y) & := & [8, 3+2x, 4+2x, 6]y \\ g(x,y) & := & \begin{bmatrix} \phi(x,y) \\ -\phi(x,y) \\ -y \end{bmatrix} \text{ with } \phi(x,y) := \begin{bmatrix} y_1+y_2-1 \\ y_3+y_4-1 \end{bmatrix} \end{array}$$

Comment: The best known values of upper-level and lower-level objectives are $F(x, y) = -11$ and $f(x, y) = 34$; cf. [8]. Two possible optimal solutions are $(2.5, 0, 1, 0, 1)$ with $F = -2.5$, $f = 14$ and $(1, 0, 1, 1, 0)$ with $F = -2$, $f = 11$.

3.2. **Linear bilevel examples.**

Problem name: `AnandalinghamWhite1990`

Source: [58]

Description: AnandalinghamWhite1990 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & -x-3y \\ G(x,y) & := & -x \\ f(x,y) & := & -x+3y \\ g(x,y) & := & \begin{bmatrix} -x-2y+10 \\ x-2y-6 \\ 2x-y-21 \\ x+2y-38 \\ -x+2y-18 \\ -y \end{bmatrix} \end{array}$$

Comment: The global optimal solution of the problem is $(16, 11)$; cf. [58].

Problem name: `Bard1984a`

Source: [75]

Description: Bard1984a is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x+y \\ G(x,y) & := & -x \\ f(x,y) & := & -5x-y \\ g(x,y) & := & \begin{bmatrix} -x-0.5y+2 \\ -0.25x+y-2 \\ x+0.5y-8 \\ x-2y-4 \\ -y \end{bmatrix} \end{array}$$

Comment: The global optimal solution is $(8/9, 20/9)$; cf. [75].

Problem name: `Bard1984b`

Source: [75]

Description: Bard1984b is defined as follows

$$
\begin{aligned}
F(x,y) &:= -5x - y \\
G(x,y) &:= -x \\
f(x,y) &:= y \\
g(x,y) &:= \begin{bmatrix} -x-0.5y+2 \\ -0.25x+y-2 \\ x+0.5y-8 \\ x-2y-4 \\ -y \end{bmatrix}
\end{aligned}
$$

Comment: The reported optimal solution is $(7.2, 1.6)$; cf. [75].

Problem name: `Bard1991Ex2`

Source: [5]

Description: Bard1991Ex2 is defined as follows

$$
\begin{aligned}
F(x,y) &:= -x + 10y_1 - y_2 \\
G(x,y) &:= -x \\
f(x,y) &:= -y_1 - y_2 \\
g(x,y) &:= \begin{bmatrix} x+y_1-1 \\ x+y_2-1 \\ y_1+y_2-1 \\ -y \end{bmatrix}
\end{aligned}
$$

Comment: The global optimal solution is $(0, 0, 1)$; cf. [5].

Problem name: `BardFalk1982Ex2`

Source: [59]

Description: BarkFalk1982Ex2 is defined as follows

$$
\begin{aligned}
F(x,y) &:= -2x_1 + x_2 + 0.5y_1 \\
G(x,y) &:= -x \\
f(x,y) &:= x_1 + x_2 - 4y_1 + y_2 \\
g(x,y) &:= \begin{bmatrix} -2x_1+y_1-y_2+2.5 \\ x_1-3x_2+y_2-2 \\ x_1+x_2-2 \\ -y \end{bmatrix}
\end{aligned}
$$

Comment: Authors in [59] claimed an optimal solution is $(1, 0, 0.5, 1)$; while authors in [79] stated the global optimal should be $(2, 0, 1.5, 0)$. And the latter is the correct result.

Problem name: `BenAyedBlair1990a`

Source: [76]

Description: BenAyedBlair1990a is defined as follows

$$
\begin{aligned}
F(x,y) &:= -1.5x - 6y_1 - y_2 \\
G(x,y) &:= \begin{bmatrix} -x \\ x-1 \end{bmatrix} \\
f(x,y) &:= -y_1 - 5y_2 \\
g(x,y) &:= \begin{bmatrix} x+3y_1+y_2-5 \\ 2x+y_1+3y_2-5 \\ -y \end{bmatrix}
\end{aligned}
$$

Comment: The global optimal solution is $(1, 0, 1)$; cf. [76].

Problem name: `BenAyedBlair1990b`
Source: [76]
Description: Ben-AyedBlair1990b is defined as follows

$$\begin{aligned} F(x,y) &:= -x-y \\ G(x,y) &:= -x \\ f(x,y) &:= y \\ g(x,y) &:= \begin{bmatrix} -4x-3y+19 \\ x+2y-11 \\ 3x+y-13 \\ -y \end{bmatrix} \end{aligned}$$

Comment: The global optimal solution is $(1, 5)$; cf. [76].

Problem name: `BialasKarwan1984a`
Source: [77]
Description: BialasKarwan1984a is defined as follows

$$\begin{aligned} F(x,y) &:= -x-y_2 \\ G(x,y) &:= -x \\ f(x,y) &:= -y_2 \\ g(x,y) &:= \begin{bmatrix} x+y_1+y_2-3 \\ -x-y_1+y_2+1 \\ -x+y_1+y_2-1 \\ x-y_1+y_2-1 \\ y_2-0.5 \\ -y \end{bmatrix} \end{aligned}$$

Comment: The global optimal solution should be $(1.5, 1, 0.5)$.

Problem name: `BialasKarwan1984b`
Source: [77]
Description: BialasKarwan1984b is defined as follows

$$\begin{aligned} F(x,y) &:= -y \\ G(x,y) &:= -x \\ f(x,y) &:= y \\ g(x,y) &:= \begin{bmatrix} -x-2y+10 \\ x+2y-38 \\ -x+2y-18 \\ x-2y-6 \\ 2x-y-21 \\ -y \end{bmatrix} \end{aligned}$$

Comment: The global optimal solution is $(16, 11)$; cf. [77].

Problem name: `CandlerTownsley1982`
Source: [60]

Description: CandlerTownsley1982 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & -8x_1 - 4x_2 + 4y_1 - 40y_2 - 4y_3 \\ G(x,y) & := & -x \\ f(x,y) & := & x_1 + 2x_2 + y_1 + y_2 + 2y_3 \\ g(x,y) & := & \begin{bmatrix} -y_1 + y_2 + y_3 + 1 \\ 2x_1 - y_1 + 2y_2 - 0.5y_3 - 1 \\ 2x_2 + 2y_1 - y_2 - 0.5y_3 - 1 \\ -y \end{bmatrix} \end{array}$$

Comment: The gloabl optimal solution for this problem is $(0, 0.9, 0, 0.6, 0.4)$ with values of upper-level and lower-level objectives $F(x,y) = -29.2$ and $f(x,y) = 3.2$; cf. [66].

Problem name: `ClarkWesterberg1988`

Source: [61]

Description: ClarkWesterberg1988 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x - 4y \\ f(x,y) & := & y \\ g(x,y) & := & \begin{bmatrix} -2x + y \\ 2x + 5y - 108 \\ 2x - 3y + 4 \end{bmatrix} \end{array}$$

Comment: The global optimal solution is $(19, 14)$; cf. [61].

Problem name: `ClarkWesterberg1990b`

Source: [10]

Description: ClarkWesterberg1990b is defined as follows

$$\begin{array}{rcl} F(x,y) & := & -x - 3y_1 + 2y_2 \\ G(x,y) & := & \begin{bmatrix} -x \\ x - 8 \end{bmatrix} \\ f(x,y) & := & -y_1 \\ g(x,y) & := & \begin{bmatrix} -y_1 \\ y_1 - 4 \\ -2x + y_1 + 4y_2 - 16 \\ 8x + 3y_1 - 2y_2 - 48 \\ -2x + y_1 - 3y_2 + 12 \end{bmatrix} \end{array}$$

Comment: The best known optimal value for the upper-level objective function is -13 and a corresponding optimal point is $(5, 4, 2)$; cf. [10].

Problem name: `GlackinEtal2009`

Source: [62]

Description: GlackinEtal2009 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & -2x_1 + 4x_2 + 3y \\ G(x,y) & := & \begin{bmatrix} x_1 - x_2 + 1 \\ -x \end{bmatrix} \\ f(x,y) & := & -y \\ g(x,y) & := & \begin{bmatrix} x_1 + x_2 + y - 4 \\ 2x_1 + 2x_2 + y - 6 \\ -y \end{bmatrix} \end{array}$$

Comment: The global optimal solution is $(1,2,0)$; cf. [62].

Problem name: `HaurieSavardWhite1990`

Source: [78]

Description: HaurieSavardWhite1990 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x+5y \\ f(x,y) & := & -y \\ g(x,y) & := & \begin{bmatrix} -3x+2y-6 \\ 3x+4y-48 \\ 2x-5y-9 \\ -x-y+8 \end{bmatrix} \end{array}$$

Comment: The global optimal solution is $(12,3)$; cf. [78].

Problem name: `HuHuangZhang2009`

Source: [63]

Description: HuHuangZhang2009 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & -4x-y_1-y_2 \\ G(x,y) & := & -x \\ f(x,y) & := & -x-3y_1 \\ g(x,y) & := & \begin{bmatrix} x+y_1+y_2-\frac{25}{9} \\ x+y_2-2 \\ y_1+y_2-\frac{8}{9} \\ -y \end{bmatrix} \end{array}$$

Comment: The global optimal solution is $\left(\frac{17}{9},\frac{8}{9},0\right)$; cf. [63]. Note that there is an error in [63] where $x+y_1-2 \leq 0$ should be $x+y_2-2 \leq 0$.

Problem name: `LanWenShihLee007`

Source: [64]

Description: LanWenShihLee2007 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & 2x-11y \\ G(x,y) & := & -x \\ f(x,y) & := & x+3y \\ g(x,y) & := & \begin{bmatrix} x-2y-4 \\ 2x-y-24 \\ 3x+4y-96 \\ x+7y-126 \\ -4x+5y-65 \\ -x-4y+8 \\ -y \end{bmatrix} \end{array}$$

Comment: The best known solution is $(17.4500, 10.9080)$; cf. [64].

Problem name: `LiuHart1994`

Source: [67]

Description: LiuHart1994 is defined as follows

$$\begin{aligned} F(x,y) &:= -x-3y \\ G(x,y) &:= -x \\ f(x,y) &:= y \\ g(x,y) &:= \begin{bmatrix} -x+y-3 \\ x+2y-12 \\ 4x-y-12 \\ -y \end{bmatrix} \end{aligned}$$

Comment: The reported optimal solution is $(4,4)$; cf. [67].

Problem name: `MershaDempe2006Ex1`

Source: [65]

Description: MershaDempe2006Ex1 is defined as follows

$$\begin{aligned} F(x,y) &:= x-8y \\ G(x,y) &:= -x \\ f(x,y) &:= y \\ g(x,y) &:= \begin{bmatrix} 5x-2y-33 \\ -x-2y+9 \\ -7x+3y-5 \\ x+y-15 \\ -y \end{bmatrix} \end{aligned}$$

Comment: The original problem has no optimal solution, so the version used here is shifted the upper level constraints to the lower level; cf. [65].

Problem name: `MershaDempe2006Ex2`

Source: [65]

Description: MershaDempe2006Ex2 is defined as follows

$$\begin{aligned} F(x,y) &:= -x-2y \\ G(x,y) &:= \begin{bmatrix} -2x+3y-12 \\ x+y-14 \end{bmatrix} \\ f(x,y) &:= -y \\ g(x,y) &:= \begin{bmatrix} -3x+y+3 \\ 3x+y-30 \end{bmatrix} \end{aligned}$$

Comment: Reported global optimal solution is $(8,6)$; cf. [65].

Problem name: `TuyEtal1993`

Source: [79]

Description: TuyEtal1993 is defined as follows

$$\begin{aligned} F(x,y) &:= -2x_1+x_2+0.5y_1 \\ G(x,y) &:= \begin{bmatrix} x_1+x_2-2 \\ -x \end{bmatrix} \\ f(x,y) &:= -4y_1+y_2 \\ g(x,y) &:= \begin{bmatrix} -2x_1+y_1-y_2+2.5 \\ x_1-3x_2+y_2-2 \\ -y \end{bmatrix} \end{aligned}$$

Comment: The reported optimal solution is $(2, 0, 1.5, 0)$; cf. [79].

Problem name: `TuyEtal1994`

Source: [80]

Description: TuyEtal1994 is defined as follows

$$\begin{aligned}
F(x,y) &:= 3x_1 + 2x_2 + y_1 + y_2 \\
G(x,y) &:= \begin{bmatrix} x_1 + x_2 + y_1 + y_2 - 4 \\ -x \end{bmatrix} \\
f(x,y) &:= 4y_1 + y_2 \\
g(x,y) &:= \begin{bmatrix} -3x_1 - 5x_2 - 6y_1 - 2y_2 + 15 \\ -y \end{bmatrix}
\end{aligned}$$

Comment: The reported global optimal solution is $(0, 3, 0, 0)$; cf. [80].

Problem name: `TuyEtal2007Ex3`

Source: [52]

Description: TuyEtal2007Ex3 is defined as follows

$$\begin{aligned}
F(x,y) &:= (12, -1, -12, 13, 0, 2, 0, -5, 6, -11)x - (5, 6, 4, 7, 0, 0)y \\
G(x,y) &:= \begin{bmatrix} -Ax - By + [-30, 134]^\top \\ -x \end{bmatrix} \\
f(x,y) &:= [3, -2, -3, -3, 1, 6]y \\
g(x,y) &:= \begin{bmatrix} -Cx - Dy + [-83, -92, -168, 96, 133, -89, 192]^\top \\ -y \end{bmatrix}
\end{aligned}$$

with A, B, C, and D respectively defined by

$$A := \begin{bmatrix} 2 & 3 & -14 & 2 & 9 & -2 & -1 & 4 & 0 & -2 \\ -1 & 7 & -13 & 0 & 15 & -2 & 8 & 4 & -4 & 7 \end{bmatrix}$$

$$B := \begin{bmatrix} 3 & -9 & 2 & 8 & -1 & 8 \\ 6 & 2 & -6 & -2 & -8 & 4 \end{bmatrix}$$

$$C := \begin{bmatrix} 5 & -7 & -4 & 2 & -3 & 9 & -9 & 1 & 3 & -11 \\ -6 & 5 & 3 & 2 & -8 & -5 & -8 & 3 & -7 & -3 \\ 6 & 4 & -2 & 0 & 2 & -3 & 3 & -2 & -2 & -4 \\ -5 & -6 & 0 & 4 & -3 & 8 & -1 & 0 & -2 & 3 \\ -11 & 11 & -4 & -5 & 10 & 6 & -14 & 7 & 11 & 3 \\ -9 & 12 & 4 & 10 & -2 & -8 & -5 & 11 & 4 & -1 \\ -7 & 2 & 6 & 0 & 11 & -1 & 2 & 2 & 1 & 2 \end{bmatrix}$$

$$D := \begin{bmatrix} -10 & 9 & 6 & -4 & -6 & 3 \\ 5 & 7 & -1 & -1 & 6 & -4 \\ -10 & -5 & -6 & 4 & -3 & 1 \\ 4 & 3 & 4 & 4 & -1 & -1 \\ 10 & 7 & -7 & -7 & -2 & -7 \\ -2 & 5 & -10 & -1 & -4 & -5 \\ 5 & 5 & 6 & 5 & -1 & 12 \end{bmatrix}$$

Comment: According to [52] the best known solution of the problem is

$$\begin{aligned}
x^* &= (0, 8.170692, 10, 0, 7.278940, 3.042311, 0, 10, 0.001982, 9.989153) \\
y^* &= (3.101280, 10, 10, 10, 0, 9.846133)
\end{aligned}$$

with $F = -467.4613, f = -11.6194$.

Problem name: `VisweswaranEtal1996`

Source: [68]

Description: VisweswaranEtal1996 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x+y \\ G(x,y) & := & -x \\ f(x,y) & := & -5x-y \\ g(x,y) & := & \begin{bmatrix} -x-0.5y+2 \\ -0.25x+y-2 \\ x+0.5y-8 \\ x-2y-2 \\ -y \end{bmatrix} \end{array}$$

Comment: The reported optimal solution is $(8/9, 20/9)$; cf. [68].

Problem name: `WangJiaoLi2005`

Source: [66]

Description: WangJiaoLi2005 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & -100x-1000y_1 \\ G(x,y) & := & \begin{bmatrix} -x \\ x-1 \end{bmatrix} \\ f(x,y) & := & -y_1-y_2 \\ g(x,y) & := & \begin{bmatrix} x+y_1-y_2-1 \\ y_1+y_2-1 \\ -y \end{bmatrix} \end{array}$$

Comment: The reported optimal values of the objective functions are $F(x,y) = -1000$ and $f(x,y) = -1$; cf. [66].

3.3. Simple bilevel examples.

Problem name: `FrankeEtal2018Ex53`

Source: [73]

Description: FrankeEtal2018Ex53 examples are defined as follows

$$\begin{array}{rcl} F(y) & := & y_1^2+y_2^2 \\ G(y) & := & \begin{bmatrix} -y \\ y-1_2 \end{bmatrix} \\ f(y) & := & (y_1-2)^2 \\ g(y) & := & \begin{bmatrix} -y \\ y-1_2 \end{bmatrix} \end{array}$$

Comment: The reported global optimal solution is $(1, 0)$; cf. [73].

Problem name: `FrankeEtal2018Ex511`

Source: [73]

Description: FrankeEtal2018Ex511 examples are defined as follows

$$\begin{array}{rcl} F(y) & := & 0.5(y_1-2)^2+0.5y_2^2+0.5(y_3-2)^2 \\ f(y) & := & y_1+y_2+y_3 \\ g(y) & := & \begin{bmatrix} -y_1-y_2 \\ -y_1+y_2 \\ -y_1 \\ -y_3 \end{bmatrix} \end{array}$$

Comment: The reported global optimal solution is $(1,-1,0)$; cf. [73].

Problem name: `FrankeEtal2018Ex513`

Source: [73]

Description: FrankeEtal2018Ex513 examples are defined as follows

$$\begin{array}{rcl} F(y) & := & -y_2 \\ f(y) & := & y_3 \\ g(y) & := & \begin{bmatrix} y_1^2-y_3 \\ y_1^2+y_2^2-1 \\ -y_3 \end{bmatrix} \end{array}$$

Comment: The reported global optimal solution is $(0,1,0)$; cf. [73].

Problem name: `FrankeEtal2018Ex521`

Source: [73]

Description: FrankeEtal2018Ex521 examples are defined as follows

$$\begin{array}{rcl} F(y) & := & -y_2 \\ f(y) & := & y_1 \\ g(y) & := & \begin{bmatrix} (y_1-1)^2-(y_2-0.5)^2-1.25 \\ y_1+y_2-1 \\ -y_1 \end{bmatrix} \end{array}$$

Comment: The reported global optimal solution is $(0,1)$; cf. [73].

Problem name: `MitsosBarton2006Ex31`

Source: [39]

Description: MitsosBarton2006Ex31 is defined as follows

$$\begin{array}{rcl} F(y) & := & y \\ G(y) & := & \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \\ f(y) & := & -y \\ g(y) & := & \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{array}$$

Comment: The optimal solution is 1; cf. [39].

Problem name: `MitsosBarton2006Ex32`

Source: [39]

Description: MitsosBarton2006Ex32 is defined as follows

$$\begin{aligned} F(y) &:= y \\ G(y) &:= \begin{bmatrix} -y-1 \\ y-1 \\ y \end{bmatrix} \\ f(y) &:= -y \\ g(y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: The problem has no optimal solution; cf. [39].

Problem name: `MitsosBarton2006Ex33`

Source: [39]

Description: MitsosBarton2006Ex33 is defined as follows

$$\begin{aligned} F(y) &:= y \\ G(y) &:= \begin{bmatrix} -y-10 \\ y-10 \end{bmatrix} \\ f(y) &:= y^2 \\ g(y) &:= \begin{bmatrix} 1-y^2 \\ -y-10 \\ y-10 \end{bmatrix} \end{aligned}$$

Comment: The optimal solution is -1; cf. [39].

Problem name: `MitsosBarton2006Ex34`

Source: [39]

Description: MitsosBarton2006Ex34 is defined as follows

$$\begin{aligned} F(y) &:= y \\ G(y) &:= \begin{bmatrix} -y-0.5 \\ y-1 \end{bmatrix} \\ f(y) &:= -y^2 \\ g(y) &:= \begin{bmatrix} -y-0.5 \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: The optimal solution is 1; cf. [39].

Problem name: `MitsosBarton2006Ex35`

Source: [39]

Description: MitsosBarton2006Ex35 is defined as follows

$$\begin{aligned} F(y) &:= y \\ G(y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \\ f(y) &:= 16y^4 + 2y^3 - 8y^2 - 1.5y + 0.5 \\ g(y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: The optimal solution is 0.5; cf. [39].

Problem name: `MitsosBarton2006Ex36`

Source: [39]
Description: MitsosBarton2006Ex36 is defined as follows

$$\begin{aligned} F(y) &:= y \\ G(y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \\ f(y) &:= y^3 \\ g(y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{aligned}$$

The optimal solution is -1; cf. [39].

Problem name: `ShehuEtal2019Ex42`
Source: [72]
Description: ShehuEtal2019Ex42 examples are defined as follows

$$\begin{aligned} F(y) &:= 0.5y^T Qy \\ f(y) &:= 0.5\|Ay-b\|_2^2 + \mu\|y\|_1 \end{aligned}$$

Comment1: Q is a positive definite matrix. $A \in \mathbb{R}^{m\times n_y}$ is a given matrix, b is a given vector and μ (e.g., 0.5) is positive scalar. b is generated as $b = Ay + \epsilon\zeta$, where A and ζ are random matrices whose elements are normally distributed with zero mean and variance 1, ϵ (e.g., 0.01) is a noisy factor. y is a generated sparse vector with few non-zero elements.

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